Generalizations on Active-Passive Dynamic Consensus Filters*

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Abstract—Active-passive dynamic consensus filters consist of agents subject to local observations of a process of interest (active agents) and agents without any observations (passive agents). In this paper, we introduce a new class of active-passive dynamic consensus filters using results from graph theory and systems science. Specifically, the proposed filters only require agents to exchange their current state information with neighbors in a simple and isotropic manner to reduce the overall information exchange cost of the network. In addition, we allow the roles of active and passive agents to be time-varying for making these filters suitable for a wide range of multiagent systems applications. We show that the proposed active-passive dynamic consensus filters enable the states of all agents to converge to an adjustable neighborhood of the average of the observations sensed by a time-varying set of active agents.

I. INTRODUCTION

For multiagent systems applications, where each agent is active in the sense that it is subject to an observation of a process of interest, existing results in dynamic consensus filters can be used [1], [2], [3], [4], [5], [6]. However, an agent can be passive for certain time instants from a practical standpoint due to its heterogeneous sensing capability in that it may not be able to sense the process and collect information. To address this problem, authors of [7], [8], [9], [10], [11] propose notable contributions. While the authors of [7], [8], [9] present methods that cover specific applications where a portion of the networked nodes are passive (and the remaining of nodes are active), their results are in the context of static consensus; that is, their distributed control algorithms are not suitable for dynamic environments.

More recently, the authors of [10], [11] propose active-passive dynamic consensus filters, where these filters not only enable the states of all agents to converge to the average of the observations only sensed by the active agents but are also suitable for multiagent systems applications in dynamic environments. However, the results in [10], [11], which utilize integral action-based distributed control algorithms, require each agent to exchange both its current state and integral state informations with its neighbors. In addition, these results assume fixed roles for active and passive agents, where this can be impractical for applications when roles of active and passive agents change with respect to time.

In this paper, we introduce a new class of active-passive dynamic consensus filters using results from graph theory and systems science. Specifically, the proposed filters only require agents to exchange their current state information with neighbors in a simple and isotropic manner to reduce the overall information exchange cost of the network. In addition, we allow the roles of active and passive agents to be time-varying for making these filters suitable for a wide range of multiagent systems applications. We show that the proposed active-passive dynamic consensus filters enable the states of all agents to converge to an adjustable neighborhood of the average of the observations sensed by a time-varying set of active agents.

The notations used in this paper are fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{R}_+^{n\times n}$ (resp., $\mathbb{R}_+^{n\times m}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative-definite) real matrices, $\mathbb{S}_+^{n\times n}$ (resp., $\mathbb{S}_+^{n\times m}$) denotes the set of $n \times n$ symmetric positive-definite (resp., symmetric nonnegative-definite) real matrices, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_+$ (resp., $\mathbb{Z}_+^n$) denotes the set of positive (resp., nonnegative) integers, $0_n$ denotes the $n \times 1$ vector of all zeros, $1_n$ denotes the $n \times 1$ vector of all ones, $0_{n\times n}$ denotes the $n \times n$ zero matrix, and $I_n$ denotes the $n \times n$ identity matrix. In addition, we write $(\cdot)^T$ for...
transpose, $(\cdot)^{-1}$ for inverse, $(\cdot)^\dagger$ for generalized inverse, $\| \cdot \|_2$ for the Euclidean norm, $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix $A$, $\lambda_i(A)$ for the $i$-th eigenvalue of $A$ ($A$ is symmetric and the eigenvalues are ordered from least to greatest value), diag$(a)$ for the diagonal matrix with the vector $a$ on its diagonal, and $[A]_{ij}$ for the entry of the matrix $A$ on the $i$-th row and $j$-th column.

II. PRELIMINARIES ON GRAPH THEORY AND ACTIVE-PASSIVE DYNAMIC CONSENSUS FILTERS

First, we recall some basic notions from graph theory (see [12], [13] for further details). In the multiagent systems literature, graphs are broadly adopted to encode interactions in networked systems. An undirected graph $G$ is defined by a set $V_G = \{1, \ldots, n\}$ of nodes and a set $E_G \subset V_G \times V_G$ of edges. If $(i, j) \in E_G$, then the nodes $i$ and $j$ are neighbors and the neighboring relation is indicated with $i \sim j$. The degree of a node is given by the number of its neighbors. Letting $d_i$ be the degree of node $i$, then the degree matrix of a graph $G$, $\mathcal{D}(G) \in \mathbb{R}^{n \times n}$, is given by

$$\mathcal{D}(G) \triangleq \text{diag}(d), \quad d = [d_1, \ldots, d_n]^T. \quad (1)$$

A path $i_0i_1 \ldots i_L$ is a finite sequence of nodes such that $i_{k-1} \sim i_k$, $k = 1, \ldots, L$, and a graph $G$ is connected if there is a path between any pair of distinct nodes. We write $\mathcal{A}(G) \in \mathbb{R}^{n \times n}$ for the adjacency matrix of a graph $G$ defined by

$$[\mathcal{A}(G)]_{ij} \triangleq \begin{cases} 1, & \text{if } (i, j) \in E_G, \\ 0, & \text{otherwise}, \end{cases} \quad (2)$$

and $\mathcal{B}(G) \in \mathbb{R}^{n \times m}$ for the incidence matrix of a graph $G$ defined by

$$[\mathcal{B}(G)]_{ij} \triangleq \begin{cases} -1, & \text{if node } i \text{ is the tail of edge } j, \\ 1, & \text{if node } i \text{ is the head of edge } j, \\ 0, & \text{otherwise}, \end{cases} \quad (3)$$

where $m$ is the number of edges, $i$ is an index for the node set, and $j$ is an index for the edge set (under the assumption that directions have been arbitrarily assigned to label the edges). By definition, note that

$$\mathcal{B}^T(G) \mathbf{1}_n = \mathbf{0}_n. \quad (4)$$

The graph Laplacian matrix, denoted by $\mathcal{L}(G) \in \mathbb{R}_+^{n \times n}$, is defined by

$$\mathcal{L}(G) \triangleq \mathcal{D}(G) - \mathcal{A}(G), \quad (5)$$

or equivalently,

$$\mathcal{L}(G) \triangleq \mathcal{B}(G) \mathcal{B}^T(G). \quad (6)$$

Throughout this paper, we model a given multiagent system by a connected, undirected graph $G$, where nodes and edges represent agents and interagent communication links, respectively.

Next, we present several necessary lemmas used in this paper.

**Lemma 1** ([12]). The spectrum of $\mathcal{L}(G)$ for a connected, undirected graph can be ordered as

$$0 = \lambda_1(\mathcal{L}(G)) < \lambda_2(\mathcal{L}(G)) \leq \cdots \leq \lambda_n(\mathcal{L}(G)), \quad (7)$$

with $\mathbf{1}_n$ as the eigenvector corresponding to the zero eigenvalue and

$$\mathcal{L}(G) \mathbf{1}_n = \mathbf{0}_n, \quad (8)$$

$$e^{\mathcal{L}(G)} \mathbf{1}_n = \mathbf{1}_n. \quad (9)$$

**Lemma 2** ([11]). Let $K = \text{diag}(k)$, $k = [k_1, k_2, \ldots, k_n]^T$, $k_i \in \mathbb{R}_+$, $i = 1, \ldots, n$, and assume that at least one element of $k$ is nonzero. Then, for a connected, undirected graph,

$$\mathcal{F}(G) \triangleq \mathcal{L}(G) + K \in \mathbb{R}_+^{n \times n}, \quad (10)$$

holds and

$$\det(\mathcal{F}(G)) \neq 0. \quad (11)$$

**Lemma 3** ([14]). For a connected, undirected graph,

$$\mathcal{L}(G) \mathcal{L}^T(G) = \mathbf{I}_n - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n^T, \quad (12)$$

holds.

Finally, we overview the active-passive dynamic consensus filter architecture introduced in [10], [11], where we refer the reader to these references for further details. Specifically, consider a system of $n$ agents exchanging information among each other using their local measurements according to a connected, undirected graph $G$. In addition, consider that there exists $m \geq 1$ inputs that interact with this system in the sense that they represent local observations of agents.

**Definition 1.** If agent $i$, $i = 1, \ldots, n$, is subject to one or more inputs (resp., no inputs), then it is an active agent (resp., passive agent).

**Definition 2.** If an input interacts with only one agent (resp., multiple agents), then it is an isolated input (resp., nonisolated input).

The active-passive dynamic consensus filter architecture of [10], [11] deals with the problem of driving the
states of all (active and passive) agents to the average of the applied inputs. For this purpose, the following integral action-based distributed control algorithm is proposed

\[
\dot{x}_i(t) = -\alpha \sum_{i \neq j} (x_i(t) - x_j(t)) + (\xi_i(t) - \xi_j(t)) + \beta_i x_i(t),
\]

\[
\dot{p}_i(t) = -\beta_i x_i(t) - \alpha \sum_{i \neq h} k_{ih}(t) (x_i(t) - c_h(t)),
\]

where \(x_i(t) \in \mathbb{R}\) and \(\xi_i(t) \in \mathbb{R}\) denote the state and the integral action of agent \(i, i = 1, \ldots, n\), respectively, that are exchanged with the neighbors of this agent, \(c_h(t) \in \mathbb{R}, h = 1, \ldots, m\), denotes an input sensed by this agent, \(\alpha \in \mathbb{R}_+\), and \(\beta_i \in \mathbb{R}_+\). Note that \(i \sim h\) notation indicates the exogenous inputs that an agent is subject to, which is similar to the \(i \sim j\) notation indicating the neighboring relation between agents.

The results in [10], [11] show that the states of all agents converge to (resp., converge to an adjustable neighborhood of) the average of the constant (resp., time-varying) inputs applied to active agents under the assumption that the roles of active and passive agents are fixed and does not change with respect to time.

### III. Reduced Information Exchange and Time-Varying Agent Roles

In this section, we propose a new class of active-passive dynamic consensus filters that only require agents to exchange their current state information with neighbors in a simple and isotropic manner to reduce the overall information exchange cost of the network, which also allows the roles of active and passive agents to be time-varying. Specifically, we propose the integral action-based distributed control algorithm given by

\[
\dot{x}_i(t) = -\alpha \sum_{i \neq j} (x_i(t) - x_j(t)) + \beta_i x_i(t) + p_i(t)
\]

\[= e^{-\gamma \sigma t} p_i(0) - \alpha \sum_{i \neq h} k_{ih}(t) (x_i(t) - c_h(t)), \]

\[x_i(0) = x_{i0}, \]

\[
\dot{p}_i(t) = -\gamma \sum_{i \neq j} (x_i(t) - x_j(t)) + \sigma p_i(t),
\]

\[p_i(0) = p_{i0}, \]

where \(x_i(t) \in \mathbb{R}\) and \(p_i(t) \in \mathbb{R}\) denote the state and the integral action of agent \(i, i = 1, \ldots, n\), respectively, with \(x_i(t)\) being the only information exchanged with the neighbors of this agent, \(c_h(t) \in \mathbb{R}, h = 1, 2, \ldots, m\) denotes an input sensed by this agent, \(\alpha \in \mathbb{R}_+, \gamma \in \mathbb{R}_+, \sigma \in \mathbb{R}_+\), and \(\beta_i \in \mathbb{R}_+\). Here, we require that there exists at least one positive \(\beta_i, i = 1, \ldots, n\). In (15), \(k_{ih}(t)\) denotes a smooth function varying on the interval \([0, 1]\). Note that if \(p_{i0} = 0\) for all \(i = 1, \ldots, n\), then one does not need the term “\(e^{-\gamma \sigma t} p_i(0)\)” in the implementation of (15).

Next, let

\[
x(t) \triangleq [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathbb{R}^n,
\]

\[
p(t) \triangleq [p_1(t), p_2(t), \ldots, p_n(t)]^T \in \mathbb{R}^n,
\]

\[
c(t) \triangleq [c_1(t), c_2(t), \ldots, c_m(t), 0, \ldots, 0]^T \in \mathbb{R}^n,
\]

where \(m \leq n\) is assumed to ease notation without loss of generality. We can now rewrite (15) and (16) in the compact form given by

\[
\dot{x}(t) = -\alpha \mathcal{L}(\mathcal{G}) x(t) - \alpha K_1(t)x(t) - \alpha \beta \mathcal{P}(t)
\]

\[+ \mathcal{P}(t) e^{\gamma \sigma t} p_0(t) \]

\[+ \alpha K_2(t) c(t), \quad x(0) = x_0,
\]

\[
\dot{p}(t) = -\gamma \mathcal{L}(\mathcal{G}) x(t) - \gamma \mathcal{P}(t), \quad p(0) = p_0,
\]

where \(\mathcal{L}(\mathcal{G}) \in \mathbb{R}^{n \times n}_+\) satisfies Lemma 1.

\[
\beta \triangleq \mathrm{diag}([\beta_1, \beta_2, \ldots, \beta_n]^T) \in \mathbb{R}^{n \times n}_+,
\]

\[
K_1(t) \triangleq \mathrm{diag}([k_{1,1}(t), \ldots, k_{1,n}(t)]^T) \in \mathbb{R}^{n \times n}_+,
\]

with

\[
k_{i,1}(t) \triangleq \sum_{i \neq h} k_{ih}(t) \in \mathbb{R}_+,
\]

denoting the number of the inputs applied to agent \(i, i = 1, \ldots, n\), and

\[
K_2(t) \triangleq \begin{bmatrix}
k_{2,11}(t) & \cdots & k_{2,1n}(t) \\
k_{2,21}(t) & \cdots & k_{2,2n}(t) \\
\vdots & \ddots & \vdots \\
k_{2,n1}(t) & \cdots & k_{2,nn}(t)
\end{bmatrix} \in \mathbb{R}^{n \times n},
\]

with

\[
k_{i,j}(t) = \sum_{j=1}^n k_{2,ij}(t).
\]

We refer the reader to [10], [11] for specific examples illustrating the construction of \(K_1(t)\) and \(K_2(t)\) matrices.

Since we are interested in driving the states of all (active and passive) agents to an adjustable neighborhood of the average of the inputs applied to the active agents, we define

\[
\delta(t) \triangleq x(t) - \epsilon(t) \mathbf{1}_n \in \mathbb{R}^n,
\]

\[
\epsilon(t) \triangleq \mathbf{1}_n^T K_2(t) c(t) \in \mathbb{R},
\]

\[
1_n^T K_2(t) \mathbf{1}_n \in \mathbb{R}.
\]
where $\delta(t)$ is the error between $x_i(t), i = 1, \ldots, n$, and the average of the applied inputs $\epsilon(t)$. Using (28), $\epsilon(t)$ can be equivalently written as

$$
\epsilon(t) = (k_{2,11}(t)c_1(t) + k_{2,12}(t)c_2(t) + \cdots + k_{2,21}(t)c_1(t) + k_{2,22}(t)c_2(t) + \cdots) / (k_{2,11}(t) + k_{2,12}(t) + \cdots + k_{2,21}(t) + k_{2,22}(t) + \cdots), \tag{29}
$$

that clearly shows the average of the applied inputs. Note that the denominator of (29) is nonzero, since we assume there exists $m \geq 1$ inputs, and hence, there exists at least one nonzero value on the denominator of (29).

Next, note that (21) can be rewritten as

$$
p(t) = e^{-\gamma t}p_0 + \int_0^t -\gamma e^{-\gamma(t-\tau)}L(G)x(\tau)d\tau = I_ne^{-\gamma t}p_0 + \int_0^t -\gamma I_ne^{-\gamma(t-\tau)}L(G)x(\tau)d\tau, \tag{30}
$$

where

$$
e^{\phi I_n} = I_ne^\phi, \quad \phi \in \mathbb{R}, \tag{31}
$$
is used [15]. In addition, using (30), (20) can be rewritten as

$$
\dot{x}(t) = -\alpha L(G)x(t) - \alpha K_1(t)x(t) - \alpha \beta x(t) + \int_0^t -\gamma e^{-\gamma(t-\tau)}L(G)x(\tau)d\tau + \alpha K_2(t)c(t), \quad x(0) = x_0, \tag{32}
$$

We now define

$$
z(t) \triangleq \int_0^t -\gamma e^{-\gamma(t-\tau)}B^T(G)x(\tau)d\tau. \tag{33}
$$

Since (6) holds, then

$$
\dot{x}(t) = -\alpha L(G)x(t) - \alpha K_1(t)x(t) - \alpha \beta x(t) + B(G)z(t) + \alpha K_2(t)c(t), \quad x(0) = x_0, \tag{34}
$$

$$
\dot{z}(t) = -\gamma B^T(G)x(t) - \gamma \sigma z(t), \quad z(0) = 0. \tag{35}
$$
follows from (32) and (33).

Using (34), the time derivative of the error (27) is given by

$$
\dot{\delta}(t) = -\alpha F(G)\delta(t) - \alpha K_1(t)\delta(t) + B(G)z(t) + L_c(t)K_2(t)c(t) - \alpha \beta \epsilon(t) - I_n\epsilon(t), \tag{36}
$$

where

$$
F(G) \triangleq L(G) + \beta \in \mathbb{S}_+, \tag{37}
$$
as a direct consequence of Lemma 2 and

$$
L_c(t) = K_1(t)I_nI_n^T - I_n, \tag{38}
$$

Now, considering

$$
e(t) \triangleq z(t) - \alpha B^T(G)L(G)L_c(t)K_2(t)c(t), \tag{39}
$$

and noting from (26) that

$$
I_n^T L_c(t) = I_n^T [K_1(t)I_nI_n^T/(I_n^T K_2(t)I_n) - I_n] = 0, \tag{40}
$$

(36) can be rewritten as

$$
\dot{\delta}(t) = -\alpha F(G)\delta(t) - \alpha K_1(t)\delta(t) + B(G)e(t) - \alpha \beta \epsilon(t) - I_n\epsilon(t). \tag{41}
$$

Finally, the time derivative of (41) is given by

$$
\dot{\epsilon}(t) = -\gamma B^T(G)\delta(t) - \gamma \sigma \epsilon(t) - \alpha \gamma \sigma K_c(t)c(t), \tag{42}
$$

$$
-\alpha \dot{K}_c(t)c(t) - \alpha K_c(t)c(t), \quad \epsilon(0) = e_0, \tag{43}
$$

with

$$
K_c(t) \triangleq B^T(G)L(G)L_c(t)K_2(t). \tag{44}
$$

Finally, the closed-loop error dynamics given by (41) and (42) can be rewritten as

$$
\dot{\delta}(t) = -\alpha F(G)\delta(t) - \alpha K_1(t)\delta(t), \tag{45}
$$

$$
\dot{\epsilon}(t) = -\gamma B^T(G)\delta(t) - \gamma \sigma \epsilon(t) + s_1(t), \tag{46}
$$

where the perturbation terms are given by

$$
s_1(t) \triangleq -\alpha \beta \epsilon(t) - I_n\epsilon(t), \tag{47}
$$

$$
s_2(t) \triangleq -\alpha \gamma \sigma K_c(t)c(t) - \alpha \dot{K}_c(t)c(t) - \alpha K_c(t)c(t) \tag{48}
$$

and we also assume that

$$
\|s_1(t)\|^2 \leq s_1^2, \tag{49}
$$

$$
\|s_2(t)\|^2 \leq s_2^2, \tag{50}
$$
hold.

Considering the active–passive dynamic consensus filter given (15) and (16), where nodes exchange information using local measurements though a connected, undirected graph topology, it can be shown that the closed-loop error dynamics given by (44) and (45) are bounded. It can be also shown that if $\alpha$ and $\gamma$ are chosen such that $1/\alpha^2$ and $\alpha^2/\gamma$ are small, then the ultimate bound of $\|\delta(t)\|_2$ can be made small for $t \geq T$. 


IV. ILLUSTRATIVE NUMERICAL EXAMPLES

In this section, the efficacy of the proposed active-passive dynamic consensus filter given by (15) and (16) is illustrated in two examples. We first consider a network with 25 agents exchanging information over a connected, undirected ring graph topology, where there are 20 passive agents and 5 active agents. Specifically, each active agent is subject to an input with these inputs given by $c_1(t) = \sin(0.1t)$, $c_2(t) = \cos(0.3t)$, $c_3(t) = \cos(0.5t) + 2\sin(0.01t)$, $c_4(t) = 0.5\sin(0.2t) + 1.5\cos(0.1t)$, and $c_5(t) = 2$. Letting all agents have arbitrary initial conditions, Figure 1 shows the response of this network with the active-passive dynamic consensus filter given by (15) and (16) subject to design parameters $\alpha = 1$, $\gamma = 1$, $\sigma = 0.1\gamma^{-1}$, and $||\beta||_2 = 0.001$. We now increase $\alpha$ to 5 and $\gamma$ to 50 such that $1/\alpha^2$ and $\alpha^2/\gamma$ are small, where Figure 2 shows the response in this case such that the states of all agents track the average of the applied inputs tighter as compared with the response in Figure 1, as expected.

Next, we consider a sensor network example with 9 agents to track a dynamic target as illustrated in Figure 3. Specifically, each agent has a sensing radius, and hence, the roles of active and passive agents change with respect to time as the dynamic target moves in this planar environment. Figure 4 shows the response of the sensor network with the active-passive dynamic consensus filter given by (15) and (16) subject to design parameters $\alpha = 20$, $\gamma = 150$, $\sigma = 0.1\gamma^{-1}$, and $||\beta||_2 = 0.001$. In this figure, the sensor network reconstructs (i.e., senses) the true trajectory of the target through local information exchange, where this illustrates the efficacy of the proposed active-passive dynamic consensus filter.

V. CONCLUSION

In this paper, we proposed a new class of active-passive dynamic consensus filters. The proposed filters only require agents to exchange their current state information with neighbors in a simple and isotropic manner and, importantly, allow the roles of active and passive agents to be time-varying for making them suitable for a wider range of multiagent systems applications. Numerical examples illustrated the efficacy of the proposed approach.

REFERENCES

Fig. 3. Sensor network with 9 agents tracking a dynamic target (dots denote the agents, circles denote the sensing radius of agents, and solid line denotes the target trajectory).

Fig. 4. Response of the sensor network depicted in Figure 3 with 9 agents communicating over a connected, undirected graph topology with the active-passive dynamic consensus filter given by (15) and (16) subject to design parameters $\alpha = 20$, $\gamma = 150$, $\sigma = 0.1\gamma^{-1}$, and $\|\beta\|_2 = 0.001$.