A Robust Adaptive Control Architecture with $\mathcal{L}_\infty$ Transient and Steady-State Performance Guarantees

Tansel Yucelen and Wassim M. Haddad

Abstract— **In this paper, a new adaptive control architecture for nonlinear uncertain dynamical systems is developed to address the problem of high-gain adaptive control. Specifically, the proposed framework involves a new and novel controller architecture involving a modification term in the update law that minimizes an error criterion involving the distance between the weighted regressor vector and the weighted system error states. This modification term allows for fast adaptation without hindering system robustness. In particular, we show that the governing tracking closed-loop system error equation approximates a Hurwitz linear time-invariant dynamical system with $\mathcal{L}_\infty$ input-output signals. This key feature of our framework allows for robust stability analysis of the proposed adaptive control law using $\mathcal{L}_1$ system theory. We further show that by properly choosing the design parameters in the modification term we can guarantee a desired bandwidth of the adaptive controller, guaranteed transient closed-loop performance, and an a priori characterization of the size of the ultimate bound of the closed-loop system trajectories.**

I. Introduction

One of the fundamental problems in feedback control design is the ability of the control system to guarantee robust stability and robust performance with respect to system uncertainties in the design model. To this end, adaptive control along with robust control theory have been developed to address the problem of system uncertainty in control-system design. The fundamental differences between adaptive control design and robust control design can be traced to the modeling and treatment of system uncertainties as well as the controller architecture structures.

In particular, adaptive control is based on constant linearly parameterized system uncertainty models of a known structure but unknown variation, whereas robust control is predicated on structured and/or unstructured linear or nonlinear (possibly time-varying) operator uncertainty models consisting of bounded variation. Hence, for systems with constant real parametric uncertainties with large unknown variations, adaptive control is clearly appropriate, whereas for systems with time-varying parametric uncertainties and nonparametric uncertainties with norm bounded variations, robust control may be more suitable.

In contrast to fixed-gain robust controllers, which are predicated on a mathematical model of the system uncertainty, and which maintain specified constants within the feedback control law to *sustain* robust stability and performance over the range of system uncertainty, adaptive controllers directly or indirectly adjust feedback gains to maintain closed-loop stability and *improve* performance in the face of system uncertainties. Specifically, indirect adaptive controllers utilize parameter update laws to identify unknown system parameters and adjust feedback gains to account for system variation, whereas direct adaptive controllers directly adjust the controller gains in response to plant variation. In either case, the overall process of parameter identification and controller adjustment constitutes a nonlinear control law architecture, which makes validation and verification of guaranteed transient performance of adaptive controllers extremely challenging [1].

Although adaptive control has been used in numerous applications to achieve system performance without excessive reliance on system models, the necessity of high-gain feedback for achieving fast adaptation can be a serious limitation of adaptive controllers [2]. Specifically, in certain applications fast adaptation is required to achieve stringent tracking performance specifications in the face of large system uncertainties and abrupt changes in system dynamics. This, for example, is the case for high performance aircraft systems that can be subjected to system faults or structural damage, which can result in major changes in aerodynamic system parameters. In such situations, high-gain adaptive control is necessary in order to rapidly reduce and maintain system tracking errors. However, fast adaptation using high-gain feedback can result in high-frequency oscillations, which can excite unmodeled system dynamics resulting in system instability [2]. Hence, there exists a critical trade-off between system stability and control adaptation rate.

Virtually, all adaptive control methods developed in the literature have averted the problem of high-gain control. Notable exceptions include [3]–[5]. Specifically, the authors in [3] use a low-pass filter that effectively subverts high frequency oscillations that can occur because of fast adaptation while using a predictor model to reconstruct the reference system model. In particular, the authors in [3] develop a robust adaptive control architecture that provides sufficient conditions for stability and performance in terms of $\mathcal{L}_1$-norms of the underlying system transfer functions despite fast adaptation, leading to uniform bounds on the $\mathcal{L}_\infty$-norms of the system input-output signals. In [4], an indirect adaptive control architecture is developed using a least-squares parameter estimation scheme to adjust the nominal controller parameters, thereby reducing system modeling errors and effectively utilizing a direct adaptive control law with a slower learning rate. More recently, the authors in [5] present a model reference adaptive control architecture for fast adaptation predicated on an optimal control problem. In particular, a fast adaptation algorithm is developed using the minimization of the system tracking error to derive a direct adaptive control law.

In this paper, a new adaptive control architecture for nonlinear uncertain dynamical systems is developed to address the problem of high-gain adaptive control. Specifically, the proposed framework involves a new and novel controller architecture involving a modification term in the update law that minimizes an error criterion involving the distance between the weighted regressor vector and the weighted system error states. This modification term allows for fast adaptation without hindering system robustness. In particular, we show that the governing tracking closed-loop system error equation approximates a Hurwitz linear time-invariant dynamical system with $\mathcal{L}_\infty$ input-output signals. This key

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T. Yucelen and W. M. Haddad are with the School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA, 30332-0150, USA. E-mails: tansel@gatech.edu, wm.haddad@aerospace.gatech.edu.

feature of our framework allows for robust stability analysis of the proposed adaptive control law using $\mathcal{L}_1$ system theory. We further show that by properly choosing the design parameters in the modification term we can guarantee a desired bandwidth of the adaptive controller, guaranteed transient closed-loop performance, and an a priori characterization of the size of the ultimate bound of the closed-loop system trajectories.

II. Notation and Mathematical Preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $(\cdot)^{+}$ denotes the Moore-Penrose generalized inverse, $\|\cdot\|_2$ denotes the Euclidian norm, and $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius matrix norm. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix M, $\sigma_{\min}(M)$ (resp., $\sigma_{\max}(M)$) for the minimum (resp., maximum) singular value of the Hermitian matrix M, $\mathrm{spec}(\cdot)$ for the spectrum of a square matrix, $\mathrm{tr}(\cdot)$ for the trace operator, $\mathrm{vec}(\cdot)$ for the column stacking operator, and $(\cdot)'$ for the Fréchet derivative.

Next, we introduce several definitions and some key results that are necessary for developing the main results of this paper.

Definition 2.1: Let $A \in \mathbb{R}^{n\times m}$ and define $A^{\mathrm{L}} \triangleq (A^{\mathrm{T}}A)^{+}A^{\mathrm{T}}$. Then, $P_A \triangleq AA^{\mathrm{L}} = A(A^{\mathrm{T}}A)^{+}A^{\mathrm{T}}$ is a *projection matrix* and $P_A^{\perp} = I - P_A$ is a *complementary projection matrix*.

Note that P_A is symmetric and nonnegative define. Nonnegative-definiteness follows from noting that $\eta^{\mathrm{T}}P_A\eta = \eta^{\mathrm{T}}A(A^{\mathrm{T}}A)^{+}A^{\mathrm{T}}\eta = \tilde{\eta}^{\mathrm{T}}(A^{\mathrm{T}}A)^{+}\tilde{\eta} \geq 0$, where $\eta \in \mathbb{R}^n$ and $\tilde{\eta} \triangleq A^{\mathrm{T}}\eta$, and by using *(xxxi)* of Proposition 6.1.6 of [6]. Furthermore, note that $P_A A = A$ and $P_A^{\perp}A = 0$.

Definition 2.2: Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function given by $\phi(\theta) \triangleq \frac{(\varepsilon_\theta+1)\theta^{\mathrm{T}}\theta-\theta_{\max}^2}{\varepsilon_\theta\theta_{\max}^2}$, where $\theta_{\max} \in \mathbb{R}$ is a projection norm bound imposed on $\theta \in \mathbb{R}^n$ and $\varepsilon_\theta > 0$ is a projection tolerance bound. Then, the *projection operator* $\mathrm{Proj} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathrm{Proj}(\theta, y) \triangleq \begin{cases} y, & \text{if } \phi(\theta) < 0, \\ y, & \text{if } \phi(\theta) \geq 0 \text{ and } \phi'(\theta)y \leq 0, \\ y - \frac{\phi'^{\mathrm{T}}(\theta)\phi'(\theta)y}{\phi'(\theta)\phi'^{\mathrm{T}}(\theta)}\phi(\theta), & \\ \quad \text{if } \phi(\theta) \geq 0 \text{ and } \phi'(\theta)y > 0, \end{cases} \quad (1)$$

where $y \in \mathbb{R}^n$.

It follows from Definition 2.2 that

$$(\theta - \theta^*)^{\mathrm{T}}(\mathrm{Proj}(\theta, y) - y) \leq 0, \quad \theta^* \in \mathbb{R}^n, \quad (2)$$

holds [7]. The definition of the projection operator can be generalized to matrices as $\mathrm{Proj_m}(\Theta, Y) = \big(\mathrm{Proj}(\mathrm{col}_1(\Theta), \mathrm{col}_1(Y)), \ \ldots, \ \mathrm{Proj}(\mathrm{col}_m(\Theta), \mathrm{col}_m(Y))\big)$, where $\Theta \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{n\times m}$, and $\mathrm{col}_i(\cdot)$ denotes i-th column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n\times m}$, it follows from (2) that $\mathrm{tr}\big[(\Theta-\Theta^*)^{\mathrm{T}}(\mathrm{Proj_m}(\Theta, Y)-Y)\big] = \sum_{i=1}^{m}\big[\mathrm{col}_i(\Theta-\Theta^*)^{\mathrm{T}}\big(\mathrm{Proj}(\mathrm{col}_i(\Theta), \mathrm{col}_i(Y))-\mathrm{col}_i(Y)\big)\big] \leq 0$ holds. Throughout the paper, we assume that the projection norm bound imposed on each column of $\Theta \in \mathbb{R}^{n\times m}$ is $\theta_{\max}$.

The following lemma presents Young's inequality [6].

Lemma 2.1: Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, and let $\sigma > 0$. Then, $x^{\mathrm{T}}y \leq \sigma x^{\mathrm{T}}x + \frac{1}{4\sigma}y^{\mathrm{T}}y$.

Lemma 2.2 ([6]): Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ be such that A and $A + B$ are nonsingular. Then, $(A + B)^{-1} = \sum_{i=0}^{k} A^{-1}(-BA^{-1})^i + A^{-1}(-BA^{-1})^{k+1}(I + BA^{-1})^{-1}$, for all $k \in \overline{\mathbb{Z}}_+$.

Lemma 2.3 ([8]): Let $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$ be symmetric matrices such that $X > 0$ and $Y < 0$. Then, XY is Hurwitz.

Lemma 2.4 ([8]): Let $A \in \mathbb{R}^{n\times n}$ be Hurwitz and let $P \in \mathbb{R}^{n\times n}$ be the unique, positive-definite solution to the Lyapunov equation $0 = A^{\mathrm{T}}P + PA + R$ for a given positive-definite $R \in \mathbb{R}^{n\times n}$. Then, for a given symmetric, nonnegative definite $K \in \mathbb{R}^{n\times n}$, $A - KP$ is Hurwitz.

Next, consider the linear dynamical system given by

$$\dot{x}(t) = Ax(t) + Bw(t), \quad x(0) = x_0, \quad t \geq 0, \quad (3)$$

$$z(t) = Cx(t), \quad (4)$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, $z(t) \in \mathbb{R}^l$, $t \geq 0$, $A \in \mathbb{R}^{n\times n}$ is Hurwitz, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$. Here, $w(\cdot)$ is an input signal belonging to the class $\mathcal{L}_\infty$ of bounded Lebesgue measurable functions on $[0, \infty)$ and $z(\cdot)$ is an output signal belonging to the class $\mathcal{L}_\infty$. For the statement of the next definition, $\|\!|\cdot|\!\|_{p,q}$ denotes a signal norm with p temporal norm and q spatial norm.

Definition 2.3: Consider the linear dynamical system (3) and (4) with $w(\cdot) \in \mathcal{L}_\infty$ and $z(\cdot) \in \mathcal{L}_\infty$. Then the $\mathcal{L}_1$ *system norm* of (3) and (4) is given by the equi-induced signal norm $\|\!|\mathcal{G}|\!\|_{(\infty,2),(\infty,2)} \triangleq \sup_{w(\cdot)\in\mathcal{L}_\infty} \frac{\|\!|z|\!\|_{\infty,2}}{\|\!|w|\!\|_{\infty,2}}$, where $\mathcal{G} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ denotes the *convolution operator* of (3) and (4) given by $z(t) = \mathcal{G}[w](t) = (\mathcal{G} * w)(t) \triangleq \int_0^\infty G(t - \tau)w(\tau)\mathrm{d}\tau$, where $G(t) \triangleq Ce^{At}B$, $t \geq 0$.

For $\alpha > 0$, define the shifted impulse response function $G_\alpha : \mathbb{R} \to \mathbb{R}^{l\times m}$ by $G_\alpha(t) \triangleq Ce^{(A+\frac{\alpha}{2}I_n)t}B$, $t \geq 0$, and let $\mathcal{G}_\alpha$ denote its convolution operator given by $z(t) = \mathcal{G}_\alpha[w](t) = (\mathcal{G}_\alpha * w)(t) \triangleq \int_0^\infty G_\alpha(t - \tau)w(\tau)\mathrm{d}\tau$. Next, we provide a computable upper bound for the $\mathcal{L}_1$ system norm $\|\!|\mathcal{G}|\!\|_{(\infty,2),(\infty,2)}$.

Theorem 2.1 ([9]): Let $\alpha > 0$ be such that $A + \frac{\alpha}{2}I$ is Hurwitz and let $Q_\alpha \in \mathbb{R}^{n\times n}$ be the unique, nonnegative definite solution to the Lyapunov equation $0 = AQ_\alpha + Q_\alpha A^{\mathrm{T}} + \alpha Q_\alpha + BB^{\mathrm{T}}$. Then, $\mathcal{G} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ and $\|\!|\mathcal{G}|\!\|_{(\infty,2),(\infty,2)} \leq \frac{1}{\sqrt{\alpha}}\sigma_{\max}^{1/2}(CQ_\alpha C^{\mathrm{T}})$.

III. Robust Adaptive Control Architecture

In this section, we introduce a new adaptive control architecture to address the problem of high-gain adaptive control. We first consider system uncertainty that is perfectly parameterized for all $x \in \mathbb{R}^n$. Then, we consider uncertainty characterizations approximated over a compact subset $\mathcal{D}_x$ of $\mathbb{R}^n$ for the case when the system uncertainty cannot be perfectly parameterized and/or the system includes bounded exogenous disturbances.

Consider the nonlinear uncertain dynamical system given by

$$\dot{x}(t) = Ax(t) + B\Delta(x(t)) + Bu(t), \ x(0) = x_0, \ t \geq 0, \quad (5)$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are known matrices such that the pair (A, B) is controllable, and $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ is a matched system uncertainty. Furthermore, we assume that the full state is available for feedback and the control input $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. In addition, consider the reference system given by

$$\dot{x}_{\mathrm{m}}(t) = A_{\mathrm{m}}x_{\mathrm{m}}(t) + B_{\mathrm{m}}r(t), \ x_{\mathrm{m}}(0) = x_0, \ t \geq 0, \quad (6)$$

where $x_{\mathrm{m}}(t) \in \mathbb{R}^n$, $t \geq 0$, is the reference state vector,

$r(t) \in \mathbb{R}^r$, $t \geq 0$, is a bounded piecewise continuous reference input, $A_\mathrm{m} \in \mathbb{R}^{n\times n}$ is Hurwitz, and $B_\mathrm{m} \in \mathbb{R}^{n\times r}$.

Assumption 3.1: The matched uncertainty in (5) is perfectly parameterized as

$$\Delta(x) = W^\mathrm{T}\beta(x), \quad x \in \mathbb{R}^n, \tag{7}$$

where $W \in \mathbb{R}^{s\times m}$ is an unknown constant weighting matrix satisfying $\|W\|_\mathrm{F} \leq w$, $w \in \mathbb{R}$ is a known positive constant, and $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is a basis function of the form $\beta(x) = [\beta_1(x),\ \beta_2(x), \ldots,\ \beta_s(x)]$ satisfying $\|\beta(x)\|_2 \leq l_{\beta 0} + l_{\beta 1}\|x\|_2$, where $l_{\beta 0} \in \mathbb{R}$ and $l_{\beta 1} \in \mathbb{R}$ are positive constants.

Remark 3.1: In general, application of standard adaptive controllers to nonlinear uncertain dynamical systems does not assume the knowledge of an upper bound w of the unknown constant weighting matrix W. That is, standard adaptive controllers can conceptually deal with uncertainties having (unrealistically) large magnitudes of W for the case where the uncertainty is perfectly parameterized. However, these controllers do not have any robustness properties, especially when the adaptation gain is chosen to be large to satisfy given performance specifications [1]. This is due to the fact that high-gain adaptive controllers involve high-bandwidths that can result in high-frequency oscillations in the closed-loop system response and excite unmodeled dynamics resulting in system instability [10]. In contrast, the design of robust adaptive controllers with large adaptation gains requires the knowledge of a *conservative* upper bound w of the unknown constant weight W (see, for example, [3]).

Remark 3.2: In Assumption 3.1, we assume that $\|\beta(x)\|_2 \leq l_{\beta 0} + l_{\beta 1}\|x\|_2$ for all $x \in \mathbb{R}^n$, which only addresses a special class of uncertainties involving a linear growth bound. Later in this section, this assumption is relaxed by addressing uncertainty parameterizations approximated over a compact set $\mathcal{D}_x$ of $\mathbb{R}^n$, similar to what is often done in the neural network literature [11]. In this case, we address a larger class of system uncertainties.

Next, consider the feedback control law given by

$$u(t) = u_\mathrm{n}(t) + u_\mathrm{ad}(t), \quad t \geq 0, \tag{8}$$

where the *nominal* control law $u_\mathrm{n}(t)$, $t \geq 0$, is given by

$$u_\mathrm{n}(t) = -K_1 x(t) + K_2 r(t), \quad t \geq 0, \tag{9}$$

and the *adaptive* control law $u_\mathrm{ad}(t)$, $t \geq 0$, is given by

$$u_\mathrm{ad}(t) = -\hat{W}^\mathrm{T}(t)\beta(x(t)), \quad t \geq 0, \tag{10}$$

where $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times r}$ are nominal control gains, and $\hat{W}(t) \in \mathbb{R}^{s\times m}$, $t \geq 0$, is an estimate of W satisfying

$$\dot{\hat{W}}(t) = \Gamma \mathrm{Proj}_\mathrm{m}\Big[\hat{W}(t),\ \beta(x(t))e^\mathrm{T}(t)PB + \gamma\Psi\big(\beta(x(t)), \hat{W}(t)\big)\Big], \quad \hat{W}(0) = \hat{W}_0, \quad t \geq 0, \tag{11}$$

where $\Gamma \in \mathbb{R}^{s\times s}$ is a positive-definite adaptation gain matrix, $e(t) \triangleq x(t) - x_\mathrm{m}(t)$, $t \geq 0$, is the system error state, $P \in \mathbb{R}^{n\times n}$ is a positive-definite solution of the Lyapunov equation

$$0 = A_\mathrm{m}^\mathrm{T} P + P A_\mathrm{m} + R, \tag{12}$$

where $R \in \mathbb{R}^{n\times n}$ is a positive-definite matrix, $\gamma > 0$ is a scalar adaptation gain, $\Psi : \mathbb{R}^s \times \mathbb{R}^{s\times m} \to \mathbb{R}^{s\times m}$ is given by

$$\Psi\big(\beta(x(t)), \hat{W}(t)\big) \triangleq -\beta(x(t))\beta^\mathrm{T}(x(t))\hat{W}(t)B^\mathrm{T} \cdot\big[\gamma A_\mathrm{m} P^{-1} + M\big]^{-\mathrm{T}} B, \ t \geq 0, \tag{13}$$

where $M \in \mathbb{R}^{n\times n}$ is a positive-definite matrix chosen such that $\det\big[\gamma A_\mathrm{m} P^{-1} + M\big] \neq 0$ and $B^\mathrm{T}\big[\gamma A_\mathrm{m} P^{-1} + M\big]^{-1} B$ is positive-definite, and $\hat{w}_\mathrm{max} \in \mathbb{R}$ is a norm bound imposed on the update weight $\hat{W}(t)$, $t \geq 0$.

For the nominal control law (9) we have the following assumption involving standard matching conditions for model reference adaptive control.

Assumption 3.2: There exist gain matrices $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times r}$ such that $A_\mathrm{m} = A - BK_1$ and $B_\mathrm{m} = BK_2$.

Remark 3.3: For system uncertainty suppression, a standard adaptive controller has the form given by ([3])

$$\dot{\hat{W}}(t) = \Gamma \mathrm{Proj}_\mathrm{m}\Big[\hat{W}(t), \beta(x(t))e^\mathrm{T}(t)PB\Big], \ \hat{W}(0) = \hat{W}_0, \quad t \geq 0. \tag{14}$$

The proposed weight update law given by (11) can be viewed as a generalization of the optimal control modification approach to the adaptive controller proposed in [5]. Specifically, if $M = A_\mathrm{m} P^{-1} - \gamma P^{-1} A_\mathrm{m}^\mathrm{T}$, then (11) specializes to the weight update law given in [5]. However, in this case, M need not be positive definite. As we see in the next sections, by properly choosing M we can guarantee a desired bandwidth of the adaptive controller, guaranteed transient closed-loop performance, and we can guarantee the size of the ultimate bound of the closed-loop system trajectories.

IV. Stability Analysis

The nonlinear uncertain dynamical system (5) can be written as

$$\dot{x}(t) = A_\mathrm{m} x(t) + B_\mathrm{m} r(t) - B\tilde{W}^\mathrm{T}(t)\beta(x(t)), \ x(0) = x_0, \quad t \geq 0, \tag{15}$$

where $\tilde{W}(t) \triangleq \hat{W}(t) - W$, $t \geq 0$, with the system error and weight update error dynamics given by, respectively,

$$\dot{e}(t) = A_\mathrm{m} e(t) - B\tilde{W}^\mathrm{T}(t)\beta(x(t)), \ e(0) = 0, \ t \geq 0, \tag{16}$$

$$\dot{\tilde{W}}(t) = \dot{\hat{W}}(t), \quad \tilde{W}(0) = \tilde{W}_0, \ t \geq 0. \tag{17}$$

We note here that since the weight update law for $\hat{W}(t)$, $t \geq 0$, is predicated on the projection operator and W is a constant matrix, there exists a norm bound $\tilde{w}_\mathrm{max}$ such that $\|\tilde{W}(t)\|_\mathrm{F} \leq \tilde{w}_\mathrm{max}$, $t \geq 0$.

Theorem 4.1 ([8]): Consider the nonlinear uncertain dynamical system given by (5) with reference system given by (6), and assume that Assumptions 3.1 and 3.2 hold. Furthermore, let the adaptive control law be given by (10) with weight update law given by (11). Then, the closed-loop error signals given by (16) and (17) are uniformly bounded for all $(e(0), \tilde{W}(0)) \in \mathcal{D}_\alpha$, where $\mathcal{D}_\alpha$ is a compact positively invariant set, with ultimate bound $\|e(t)\|_2 < \varepsilon$, $t \geq T$, where

$$\varepsilon \triangleq \Big[\lambda_\mathrm{max}(P)\vartheta^2 + \lambda_\mathrm{max}(\Gamma^{-1})\tilde{w}_\mathrm{max}^2\Big]^{\frac{1}{2}}, \tag{18}$$

$$\vartheta \triangleq \Big[\frac{c_2^2}{4c_1^2} + \frac{c_3}{c_1}\Big]^{\frac{1}{2}} + \frac{c_2}{2c_1}, \tag{19}$$

$c_1 \triangleq \lambda_\mathrm{min}(R) - \frac{1}{2}\gamma\|S^{\frac{1}{2}}\|_\mathrm{F}^2 w^2 l_{\beta 1}^2 > 0$, $c_2 \triangleq \gamma\|S^{\frac{1}{2}}\|_\mathrm{F}^2 w^2 \bar{l}_{\beta 0} l_{\beta 1}$, $c_3 \triangleq \frac{1}{2}\gamma\|S^{\frac{1}{2}}\|_\mathrm{F}^2 w^2 l_{\beta 0}^2$, $S \triangleq B^\mathrm{T}\big[\gamma A_\mathrm{m} P^{-1} + M\big]^{-\mathrm{T}} B$, $\bar{l}_{\beta 0} \triangleq l_{\beta 0} + l_{\beta 1} x_\mathrm{m}^*$, and $\|x_\mathrm{m}(t)\| \leq x_\mathrm{m}^*$, $t \geq 0$. In addition, for $t \in [0, T)$, the system error and weight update error dynamics satisfy

$$\|e(t)\|_2 \leq \|\tilde{W}(0)\|_\mathrm{F}\Big[\|\Gamma^{-1}\|_\mathrm{F}/\lambda_\mathrm{min}(P)\Big]^{\frac{1}{2}}, \tag{20}$$

$$\|\tilde{W}(t)\|_\mathrm{F} \leq \|\tilde{W}(0)\|_\mathrm{F}\Big[\|\Gamma^{-1}\|_\mathrm{F}/\lambda_\mathrm{min}(\Gamma^{-1})\Big]^{\frac{1}{2}}. \tag{21}$$

Remark 4.1: In Theorem 4.1 we assume that $\lambda_\mathrm{min}(R) - \frac{1}{2}\gamma\|S^{\frac{1}{2}}\|_\mathrm{F}^2 w^2 l_{\beta 1}^2 > 0$. Since a *conservative* upper bound w of the unknown constant weight W is known, this can be easily satisfied by judiciously choosing the design parameters. As noted in Remark 3.1, similar conditions are required to be satisfied in the design process for guaranteeing a robust adaptive controller design (see, for example, [3]).

Remark 4.2: Theorem 4.1 shows that over a transient finite-time T the closed-loop error signals (16) and (17) are bounded from above by (20) and (21), respectively. This implies along with uniform ultimate boundedness of the closed-loop error signals (16) and (17) that $e(\cdot) \in \mathcal{L}_\infty$ and vec $\tilde{W}(\cdot) \in \mathcal{L}_\infty$, and hence, $x(\cdot) \in \mathcal{L}_\infty$ and $u(\cdot) \in \mathcal{L}_\infty$. Furthermore, note that $e(t)$, $t \in [0, T)$, can be made sufficiently small (satisfying ϑ in (19)) by letting $\lambda_{\min}(\Gamma) \to \infty$.

Remark 4.3: Let $M = mI$ in (13). Since $\gamma\big[\gamma A_{\rm m}P^{-1} + mI\big]^{-1} = \big[A_{\rm m}P^{-1} + \gamma^{-1}mI\big]^{-1}$, it follows from Lemma 2.2, with $A \triangleq \gamma^{-1}mI$ and $B \triangleq A_{\rm m}P^{-1}$, that $\big[A_{\rm m}P^{-1} + \gamma^{-1}mI\big]^{-1} = 0$ as $\gamma \to 0$ or $m \to \infty$. That is, $\gamma\Psi\big(\beta(x(t)), \hat{W}(t)\big) = 0$ as $\gamma \to 0$ or $m \to \infty$. In this case, we obtain the standard adaptive controller given by (14). Furthermore, it follows from (19) that $\vartheta = 0$, that is, we recover asymptotic stability of the system error dynamics. However, $\gamma \to 0$ or $m \to \infty$ (resp., $\lambda_{\min}(M) \to \infty$) are not desirable design choices since they might lead to a high bandwidth adaptive controller that can result in high-frequency oscillations in the closed-loop system response and excite unmodeled dynamics resulting in system instability. This is discussed later below.

Next, we relax Assumption 3.1 to the case where $\|\beta(x)\|_2 \le l_{\beta 0} + l_{\beta 1}\|x\|_2$ does not hold for all $x \in \mathbb{R}^n$ and the system uncertainty $\Delta(x)$ cannot be perfectly parameterized.

Assumption 4.1: The matched uncertainty in (5) is linearly parameterized as

$$\Delta(x) \;=\; W^{\rm T}\beta(x) + \eta(x), \quad x \in \mathcal{D}_x, \tag{22}$$

where $W \in \mathbb{R}^{s\times m}$ is an unknown constant weighting matrix satisfying $\|W\|_{\rm F} \le w$, $w \in \mathbb{R}$ is a known positive constant, $\beta : \mathcal{D}_x \to \mathbb{R}^s$ is a basis function of the form $\beta(x) = [\beta_1(x), \ \beta_2(x), \ldots, \ \beta_s(x)]$ satisfying $\|\beta(x)\|_2 \le l_{\beta 0} + l_{\beta 1}\|x\|_2$, $x \in \mathcal{D}_x$, where $l_{\beta 0} \in \mathbb{R}$ and $l_{\beta 1} \in \mathbb{R}$ are positive constants, $\eta : \mathcal{D}_x \to \mathbb{R}^m$ is the system modeling error satisfying $\|\eta(x)\|_2 \le \eta_{\max}$, $\eta_{\max} > 0$, and $\mathcal{D}_x$ is a compact subset of $\mathbb{R}^n$.

Remark 4.4: Note that Assumption 4.1 does not assume boundedness of the basis function $\beta(x)$, $x \in \mathcal{D}_x$, as is standard in the neuroadaptive control literature [11]. Furthermore, in this case, a basis function constructed using neural networks is not the only choice for $\beta(x)$ since the boundedness assumption on $\beta(x)$, $x \in \mathcal{D}_x$, that is, $\|\beta(x)\|_2 \le l_{\beta 0}$, $x \in \mathcal{D}_x$, is relaxed to $\|\beta(x)\|_2 \le l_{\beta 0} + l_{\beta 1}\|x\|_2$, $x \in \mathcal{D}_x$.

Using the uncertainty parametrization given by (22), the system error dynamics are now given by

$$\dot{e}(t) \;=\; A_{\rm m}e(t) - B\tilde{W}^{\rm T}(t)\beta(x(t)) + B\eta(x(t)), \ e(0) = 0, \quad t \ge 0, \tag{23}$$

where $\tilde{W}(t)$, $t \ge 0$, is given by (17).

Theorem 4.2 ([8]): Consider the nonlinear uncertain dynamical system given by (5) with reference system given by (6), and assume that Assumptions 3.2 and 4.1 hold. Furthermore, let the adaptive control law be given by (10) with weight update law given by (11). Then, the closed-loop error signals given by (23) and (17) are uniformly bounded for all $(e(0), \tilde{W}(0)) \in \mathcal{D}_\alpha$, where $\mathcal{D}_\alpha$ is a compact positively invariant set, with ultimate bound $\|e(t)\|_2 < \bar{\varepsilon}$, $t \ge T$, where

$$\bar{\varepsilon} \;\triangleq\; \Big[\lambda_{\max}(P)\bar{\vartheta}^2 + \lambda_{\max}(\Gamma^{-1})\tilde{w}^2_{\max}\Big]^{\frac{1}{2}}, \tag{24}$$

$$\bar{\vartheta} \;\triangleq\; \left[\frac{\bar{c}_2^2}{4c_1^2} + \frac{c_3}{c_1}\right]^{\frac{1}{2}} + \frac{\bar{c}_2}{2c_1}, \tag{25}$$

$c_1 \triangleq \lambda_{\min}(R) - \frac{1}{2}\gamma\|S^{\frac{1}{2}}\|_{\rm F}^2 w^2 l_{\beta 1}^2 > 0$, $\bar{c}_2 \triangleq \gamma\|S^{\frac{1}{2}}\|_{\rm F}^2 w^2 \bar{l}_{\beta 0} l_{\beta 1} + 2\|PB\|_{\rm F}\eta_{\max}$, $c_3 \triangleq \frac{1}{2}\gamma\|S^{\frac{1}{2}}\|_{\rm F}^2 w^2 l_{\beta 0}^2$, $S \triangleq B^{\rm T}\big[\gamma A_{\rm m}P^{-1} + M\big]^{-{\rm T}}B$, $\bar{l}_{\beta 0} \triangleq l_{\beta 0} + l_{\beta 1}x^*_{\rm m}$, and $\|x_{\rm m}(t)\| \le x^*_{\rm m}$, $t \ge 0$. In addition, the system error and weight update error dynamics satisfy the bounds (20) and (21), respectively.

V. Transient and Steady-State Performance Guarantees

In this section, we show that the governing tracking closed-loop system error equation *approximates* a Hurwitz linear time-invariant dynamical system with $\mathcal{L}_\infty$ input-output signals. This allows us to derive uniform transient and steady-state performance bounds in terms of $\mathcal{L}_1$-norms of the closed-loop system error dynamics that are independent of the system adaptation rate. The following lemma is key in developing the results of this section.

Lemma 5.1 ([8]): Consider the weight update law given by (11). If $M = -\gamma A_{\rm m}P^{-1} + mI$, $m > 0$, then

$$B\hat{W}^{\rm T}(t)\beta(x(t)) \;=\; \gamma^{-1}mP_BPe(t) - \gamma^{-1}mB^{\rm L^T}\Big[\beta^{\rm L}(x(t)) \cdot\big[\Gamma^{-1}\dot{\hat{W}}(t) - \kappa(t)\big]\Big]^{\rm T}, \quad t \ge 0, \tag{26}$$

where $P_B \triangleq B(B^{\rm T}B)^+B^{\rm T}$, $B^{\rm L} \triangleq (B^{\rm T}B)^+B^{\rm T}$, $\beta^{\rm L}(x(t)) \triangleq \big(\beta^{\rm T}(x(t))\beta(x(t))\big)^+\beta(x(t))^{\rm T}$, and $\kappa(t) \in \mathbb{R}^{s\times m}$, $t \ge 0$, is a residual error signal.

Remark 5.1: The residual error $\kappa(t) \in \mathbb{R}^{s\times m}$, $t \ge 0$, in (26) satisfies $\|\kappa(t)\|_{\rm F} \le \kappa_{\max}$, $t \ge 0$, since all the signals in (26) are bounded by Theorem 4.1 (resp., Theorem 4.2). Furthermore, $\kappa(t)$, $t \ge 0$, is identically zero when $\hat{W}_{ij}(t) \in (-\hat{w}_{\max}, \hat{w}_{\max})$, for all $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$, where $\hat{w}_{\max}$ is the projection bound of $\hat{W}(t)$, $t \ge 0$. Hence, by choosing a conservative projection bound $\hat{w}_{\max}$, $\kappa(t) = 0$ for all $t \ge 0$.

Proposition 5.1 ([8]): Consider the system error dynamics given by (16) (resp., (23)) and let $M = -\gamma A_{\rm m}P^{-1} + mI$, $m > 0$. Furthermore, assume Assumption 3.1 (resp., Assumption 4.1) holds. Then, the system error dynamics (16) (resp., (23)) can be characterized by the linear, time-invariant system $\mathcal{G}_{e,w}$ given by

$$\dot{e}(t) \;=\; \tilde{A}e(t) + \tilde{B}w(t), \quad e(0) = e_0, \quad t \ge 0, \tag{27}$$

$$z(t) \;=\; \tilde{C}e(t), \tag{28}$$

where $\tilde{A} \triangleq A_{\rm m} - \gamma^{-1}mP_BP$ is Hurwitz, $\tilde{B} \triangleq B$, $\tilde{C} \triangleq I$, $w(t) \triangleq \gamma^{-1}m(B^{\rm T}B)^+\Big[\beta^{\rm L}(x(t)) \cdot\big[\Gamma^{-1}\dot{\hat{W}}(t) - \kappa(t)\big]\Big]^{\rm T} + W^{\rm T}\beta(x(t))$ $\Big($resp., $w(t) \triangleq \gamma^{-1}m(B^{\rm T}B)^+\Big[\beta^{\rm L}(x(t))\big[\Gamma^{-1}\dot{\hat{W}}(t) - \kappa(t)\big]\Big]^{\rm T} + W^{\rm T}\beta(x(t)) + \eta(x(t))\Big)$, and $w(\cdot) \in \mathcal{L}_\infty$.

Note that (27) and (28) characterizes the closed-loop linear time-invariant dynamical system. Since $w(\cdot) \in \mathcal{L}_\infty$ and $\tilde{A}$ is Hurwitz, it follows that $z(\cdot) \in \mathcal{L}_\infty$. Hence, the following theorem is immediate.

Theorem 5.1 ([8]): Consider the linear, time-invariant system $\mathcal{G}_{e,w}$ given by (27) and (28). Let $\alpha > 0$ be such that $\tilde{A} + \frac{\alpha}{2}I$ is Hurwitz and let $Q_\alpha \in \mathbb{R}^{n\times n}$ be the unique, nonnegative definite solution to the Lyapunov equation $0 = \tilde{A}Q_\alpha + Q_\alpha \tilde{A}^{\mathrm{T}} + \alpha Q_\alpha + \tilde{B}\tilde{B}^{\mathrm{T}}$. Then, $\mathcal{G}_{e,w} : \mathcal{L}_\infty \to \mathcal{L}_\infty$ and $|||\mathcal{G}_{e,d}|||_{(\infty,2),(\infty,2)} \le \frac{1}{\sqrt{\alpha}} \sigma_{\max}^{1/2}(Q_\alpha)$.

The following proposition discusses the special case when $\lambda_{\min}(\Gamma) \to \infty$ and $\hat{W}_{ij}(t) \in (-\hat{w}_{\max}, \hat{w}_{\max})$ for all $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$, where $\hat{w}_{\max}$ is the projection bound of $\hat{W}(t)$, $t \ge 0$. In this case, $\Delta(x(t))$, $t \ge 0$, in (7) (resp., in (22)) is directly related to $z(t)$, $t \ge 0$, in (28), through $\mathcal{G}_{e,w}$.

Proposition 5.2 ([8]): Consider the system error dynamics given by (16) and let $M = -\gamma A_{\mathrm{m}} P^{-1} + mI$, $m > 0$, and $\Gamma = \xi \Gamma_\xi$, $\xi > 0$, $\Gamma_\xi > 0$. Furthermore, assume Assumption 3.1 (resp. Assumption 4.1) holds and $\dot{\hat{W}}(t)$, $t \ge 0$, is bounded as $\xi \to \infty$. If $\xi \to \infty$, then the system error dynamics (16) (resp. (23)) can be characterized by the linear, time-invariant system $\mathcal{G}_{e,w}$ given by (27) and (28), where $\tilde{A} \triangleq A_{\mathrm{m}} - \gamma^{-1} m P_B P$ is Hurwitz, $\tilde{B} \triangleq B$, $\tilde{C} \triangleq I$, $w(t) \triangleq \gamma^{-1} m (B^{\mathrm{T}}B)^{+} \big[-\beta^{\mathrm{L}}(x(t))\kappa(t)\big]^{\mathrm{T}} + W^{\mathrm{T}}\beta(x(t))$ $\Big($resp., $w(t) \triangleq \gamma^{-1} m (B^{\mathrm{T}}B)^{+} \big[-\beta^{\mathrm{L}}(x(t))\kappa(t)\big]^{\mathrm{T}} + W^{\mathrm{T}}\beta(x(t)) + \eta(x(t))\Big)$, and $w(\cdot) \in \mathcal{L}_\infty$. If, in addition, $\hat{W}_{ij}(t) \in (-\hat{w}_{\max}, \hat{w}_{\max})$, for all $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$, where $\hat{w}_{\max}$ is the projection bound of $\hat{W}(t)$, $t \ge 0$, then $w(t) \triangleq \Delta(x(t))$, where $\Delta(x(t))$, $t \ge 0$, is given by (7) (resp., (22)), and $w(\cdot) \in \mathcal{L}_\infty$.

Remark 5.2: Proposition 5.2 characterizes a closed-loop linear time-invariant dynamical system from the system input $\Delta(x(t))$, $t \ge 0$, given by (7) (resp., (22)), to the system output $e(t)$, $t \ge 0$. Note that for both results, we assume the time derivative of $\hat{W}(t)$, $t \ge 0$, remains bounded as $\xi \to \infty$. This assumption is not restrictive since the modification term given by (13) can be viewed as a *variable* and *weighted damping* term due to its sign definiteness and dependence on $\hat{W}(t)$, $t \ge 0$, and hence, it limits the frequency content of the weight update law given by (11) resulting in a bounded time derivative of $\hat{W}(t)$, $t \ge 0$. This observation holds even in the face of infinitely large adaptation gains.

Remark 5.3: By choosing a conservative projection norm bound $\hat{w}_{\max}$ (see Remark 5.1) and letting $\lambda_{\min}(\Gamma) \to \infty$, the closed-loop linear time-invariant dynamical system given by (27) and (28) characterizes a mapping when $\Delta(x(t))$, $t \ge 0$, given by (7) (resp., (22)) is viewed as a system input and $e(t)$, $t \ge 0$, is viewed as a system output. Hence, closed-loop system properties such as system bandwidth, transient system performance, and $\mathcal{L}_1$ system norm of $\mathcal{G}_{e,w}$ (Theorem 5.1) can be analyzed using (27) and (28).

As noted in Remark 4.3, $\gamma \to 0$ or $m \to \infty$ (resp., $\lambda_{\min}(M) \to \infty$) are not always desirable design choices. This is because in this case $\mathcal{G}_{e,w}$ has infinitely fast decaying poles in the left-half plane, and hence, $\mathcal{G}_{e,w}$ behaves like a high-bandwidth filter, which can be very sensitive to variations in $w(t)$ causing high-frequency oscillations in the system response. Hence, by varying the parameters γ and M we can a guarantee a prescribed closed-loop system bandwidth.

VI. Connections to Gradient Minimization

Finally, we show that the proposed modification term (13) in the adaptive update law (11) can be derived from a minimization problem involving an error criterion capturing the distance between the weighted regressor vector and the weighted error states. Specifically, consider the cost function given by $\mathcal{J}(e, \hat{W}) = \frac{1}{2}\|Pe - B\hat{W}^{\mathrm{T}}\beta(x)\|_2^2$ and note that the gradient of $\mathcal{J}(e, \hat{W})$ with respect to $\hat{W}$ is given by

$$\frac{\partial \mathcal{J}(e(t), \hat{W}(t))}{\partial \hat{W}(t)} = -\beta(x(t))e^{\mathrm{T}}(t)PB + \beta(x(t))\beta^{\mathrm{T}}(x(t)) \cdot \hat{W}(t)B^{\mathrm{T}}B, \quad t \ge 0. \tag{29}$$

Now, constructing the weight update law to be the negative gradient of $\mathcal{J}(e, \hat{W})$, we obtain

$$\dot{\hat{W}}(t) = -\Gamma \frac{\partial \mathcal{J}(e(t), \hat{W}(t))}{\partial \hat{W}(t)}, \ \hat{W}(0) = \hat{W}_0, \ t \ge 0, \tag{30}$$

where $\Gamma > 0$. Hence, choosing Γ sufficiently large it follows that $\mathcal{J}(e, \hat{W})$ is minimized. If $M = -\gamma A_{\mathrm{m}} P^{-1} + I$, then (11) specializes to (30). If, in addition, $M = -\gamma A_{\mathrm{m}} P^{-1} + I$ is positive definite, then (11) can be viewed as a gradient optimization based weight update law that minimizes $\mathcal{J}(e, \hat{W})$.

VII. Disturbance Rejection Problem

The results of this paper can be easily applied to the disturbance rejection problem. To see this, replace $\Delta(x(t))$ in (5) with $d(t) \in \mathbb{R}^m$, $t \ge 0$, where $d(t)$ is an unknown disturbance satisfying $\|d(t)\|_2 \le d_{\max}$, $t \ge 0$, and $\|\dot{d}(t)\|_2 \le \dot{d}_{\max}$, $t \ge 0$. In this case, the adaptive control law in (10) becomes $u_{\mathrm{ad}}(t) = -\hat{d}(t)$, $t \ge 0$, where $\hat{d}(t) \in \mathbb{R}^m$, $t \ge 0$, is an estimate of $d(t)$, $t \ge 0$, satisfying $\dot{\hat{d}}(t) = \Gamma \mathrm{Proj}\Big[\hat{d}(t),\ B^{\mathrm{T}}Pe(t) - \gamma B^{\mathrm{T}}\big[\gamma A_{\mathrm{m}} P^{-1} + M\big]^{-1} B\hat{d}(t)\Big]$, $\hat{d}(0) = \hat{d}_0$, $t \ge 0$, with $M \in \mathbb{R}^{n \times n}$ being a positive-definite matrix chosen such that $\det\big[\gamma A_{\mathrm{m}} P^{-1} + M\big] \neq 0$ and $B^{\mathrm{T}}\big[\gamma A_{\mathrm{m}} P^{-1} + M\big]^{-1} B$ is positive definite, and $\hat{d}_{\max} \in \mathbb{R}$ being a norm bound imposed on the update weight $\hat{d}(t)$, $t \ge 0$. An identical result to Theorem 4.1 can now be established by considering the Lyapunov-like function candidate given by $V(e, \tilde{d}) = e^{\mathrm{T}}Pe + \tilde{d}\Gamma^{-1}\tilde{d}$, where $\tilde{d}(t) \triangleq \hat{d}(t) - d(t)$, $t \ge 0$. Furthermore, if we set $M = -\gamma A_{\mathrm{m}} P^{-1} + mI$, $m > 0$, in the weight update law, then using similar arguments as in Lemma 5.1 it can be shown that $B\hat{d}(t) = \gamma^{-1} m P_B Pe(t) - \gamma^{-1} m B^{\mathrm{L}^{\mathrm{T}}}\big[\Gamma^{-1}\dot{\hat{d}}(t) - \kappa(t)\big]$, $t \ge 0$, with $\kappa(t) \in \mathbb{R}^m$, $t \ge 0$, being the residual error signal. In this case, identical results to Propositions 5.1 and 5.2 and Theorem 5.1 can be easily obtained.

VIII. Illustrative Numerical Example

We present a numerical example to illustrate the proposed adaptive controller designed based on Theorem 4.1. Specifically, consider the linear dynamical system representing an uncertain aircraft rolling dynamics model given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta(x(t)) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \tag{31}$$

$x_1(0) = x_2(0) = 0$ and $t \ge 0$, where x_1 represents the roll angle in radians, x_2 represents the roll rate in radians per second, and $\Delta(x) = \big[0.2314,\ 0.7848\big]^{\mathrm{T}}$. For our simulation, we choose $K_1 = [0.25,\ 0.80]$ and $K_2 = 0.25$, and a reference system corresponding to a natural frequency $\omega_{\mathrm{n}} = 0.50$ rad/s and a damping ratio $\zeta = 0.80$. Note that Assumptions 3.2 and 3.1 hold. Note also that the nominal closed-loop system is very lightly damped without adaptation ($u_{\mathrm{ad}}(t) \equiv 0$, $t \ge 0$). We set $\beta(x) = x$, $R = I_2$, $\gamma = 1$, $M = 4I_2$, $\Gamma = 100000I_2$, and $\hat{w}_{\max} = 2$ (a conservative choice for projection bound). It follows from Theorem 5.1 that $|||\mathcal{G}_{e,w}|||_{(\infty,2),(\infty,2)} \le 0.35$ for $\alpha = 2.85$. Here, our aim is to track a given filtered square-wave roll angle reference command $r(t)$, $t \ge 0$.

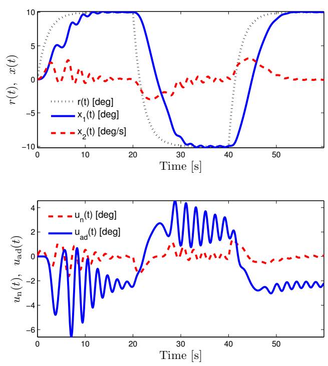

Fig. 1. Closed-loop system response of the uncertain aircraft rolling dynamics model with the standard adaptive controller for a reference command having a maximum amplitude of 10 ((14) with $\Gamma = 100I_2$).

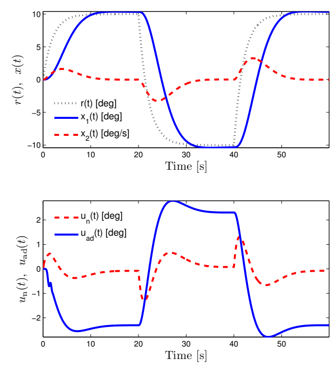

Fig. 2. Closed-loop system response of the uncertain aircraft rolling dynamics model with the proposed adaptive controller for a reference command having a maximum amplitude of 10 ((11) with $\Gamma = 100000I_2$, $\gamma = 1$, and $M = 4I_2$).

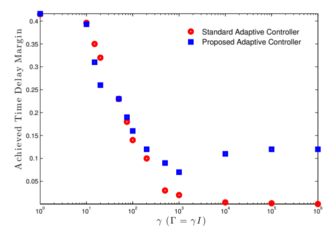

Fig. 3. Achieved time-delay margin of the uncertain aircraft rolling dynamics model with the standard and proposed adaptive controllers for a reference command having a maximum amplitude of 10.

Figure 1 shows the closed-loop system performance of a standard adaptive controller given by (14) with an adaptation gain of $\Gamma = 100I_2$ and the same projection bound for a reference command profile having a maximum amplitude of 10 degrees. It can be seen from this figure that the system response and the controller performance are oscillatory. Figure 2 shows the closed-loop system performance of the proposed adaptive controller given by (11) for the same command profile. Note that the tracking performance of the proposed controller in Figure 2 is clearly superior as compared to the standard adaptive controller in Figure 1.

This system performance is expected since the proposed robust adaptive controller does not have a high bandwidth in the face of high gain control. This implies that the proposed adaptive controller possesses robustness against input time-delays since its bandwidth is limited. To see this, we inserted a time-delay to the control input and determined the achieved time-delay margin of the uncertain aircraft rolling dynamics model with both the standard and proposed adaptive controllers. Figure 3 shows that the proposed adaptive controller's achieved time-delay margin is bounded away from zero and goes to 0.12 seconds as we increase the system adaptation gain. In contrast, the standard adaptive controller's achieved time-delay margin approaches zero as we increase the system adaptation gain. In future research, we will investigate the guaranteed robustness properties of the proposed adaptive controller against input time-delays.

IX. Conclusion

It is well known that standard model reference adaptive control methods employ high-gain feedback to achieve fast adaptation in order to rapidly reduce system tracking errors in the face of large system uncertainties. High-gain feedback, however, leads to increased controller effort and reduced relative stability margins, and can excite unmodeled dynamics and drive the system to instability. In this paper, we presented a new robust adaptive control architecture that allows for fast adaptation of nonlinear uncertain dynamical systems while guaranteeing transient and steady-state performance bounds. The proposed architecture allows for a trade-off between tracking performance and stability robustness providing a systematic framework for verification and validation of adaptive control systems. Future research will include analyzing relative stability gain and time delay margins and extensions to output feedback control for minimum and nonminimum phase uncertain systems.

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