On Model Reference Adaptive Control for Uncertain Dynamical Systems with Unmodeled Dynamics

K. Merve Dogan, Tansel Yucelen, Benjamin C. Gruenwald, and Jonathan A. Muse

Abstract— **On model reference adaptive control for uncertain dynamical systems, it is well know that there exists a fundamental stability limit, where the closed-loop dynamical system subject to this class of control laws remains stable either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. Specifically, this implies that model reference adaptive controllers cannot tolerate large system uncertainties even when unmodeled dynamics satisfy a set of conditions. Motivated from this standpoint, this paper proposes a model reference adaptive control approach to relax this fundamental stability limit, where an adaptive control signal is augmented with an adaptive robustifying term. The key feature of our framework allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled dynamics satisfy a set of conditions. An illustrative numerical example is provided to demonstrate the efficacy of the proposed approach.**

I. Introduction

Achieving system stability in the presence of not only system uncertainties but also unmodeled dynamics is a challenge in the design of model reference adaptive controllers. In particular, the authors of the seminal paper [1] investigate the stability of the closed-loop dynamical system subject to a model reference adaptive controller and reveal that the presence of unmodeled dynamics can result in system instability. Motivated from this phenomenon, several approaches are proposed in the literature (see, for example, [2]–[4] and references therein) that are aimed toward robust model reference adaptive controller designs.

Although these approaches can achieve system stability in the presence of unmodeled dynamics with respect to a set of initial conditions or under the assumption of persistency of excitation, the authors of [5]–[9] recently use a projection operator-based approach [10] to show the boundedness of the closed-loop dynamical system without making assumptions on initial conditions and persistency of excitation — when a fundamental stability limit holds for model reference adaptive controllers [5]. From a practical standpoint, this fundamental stability limit implies that the closed-loop dynamical system subject to a model reference adaptive controller remains stable either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. That is, these controllers cannot tolerate large system uncertainties even when unmodeled dynamics satisfy a set of conditions.

Considering one of the main advantages of using adaptive controllers versus fixed-gain robust controllers, which is the capability of tolerating large system uncertainty levels (in the absence of unmodeled dynamics), it is of interest to relax this fundamental stability limit to allow model reference adaptive controllers to tolerate large system uncertainty levels also in the presence of unmodeled dynamics. Motivated from this standpoint, the contribution of this paper is to present a model reference adaptive control approach with relaxed fundamental stability limit. Specifically, we propose a projection operator-based adaptive robustifying term that augments a projection operator-based adaptive control signal. The key feature of our framework allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled dynamics satisfy a set of conditions.

The organization of this paper is as follows. Section II presents the standard model reference adaptive control problem formulation in the presence of unmodeled dynamics and shows a version of the fundamental stability limit revealed in [5], where this limit is relaxed in Section III with the proposed approach to the model reference adaptive control problem. An illustrative numerical example is provided in Section IV to demonstrate the efficacy of the proposed approach. Finally, concluding remarks are summarized in Section V.

II. Mathematical Preliminaries

We first begin with the notation used throughout this paper. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes

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the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp., $\overline{\mathbb{R}}_+$) denotes the set of positive (resp., nonnegative) real numbers, $\mathbb{R}_+^{n\times n}$ (resp., $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative-definite) real matrices, $\mathbb{S}^{n\times n}$ denotes the set of $n\times n$ symmetric real matrices, $\mathbb{D}^{n\times n}$ denotes the set of $n\times n$ real matrices with diagonal scalar entries, and "$\triangleq$" denotes the equality by definition. In addition, we use $(\cdot)^{\mathrm{T}}$ for the transpose operator, $(\cdot)^{-1}$ for the inverse operator, $\mathrm{tr}(\cdot)$ for the trace operator, $\overline{\lambda}(A)$ (resp., $\underline{\lambda}(A)$) for the maximum (resp., minimum) eigenvalue of the matrix $A \in \mathbb{R}^{n\times n}$, $\|\cdot\|_2$ for the Euclidean norm, and

$$\|A\|_2 \triangleq \left(\overline{\lambda}(A^{\mathrm{T}}A)\right)^{\frac{1}{2}}, \tag{1}$$

for the induced 2-norm of the matrix $A \in \mathbb{R}^{n\times m}$.

In the reminder of this section, we present the standard model reference adaptive control problem formulation in the presence of unmodeled dynamics and show a version of the fundamental stability limit revealed in [5]. Specifically, we first give the following definition.

Definition 1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable convex function given by

$$\phi(\theta) \triangleq \frac{(\epsilon_\theta + 1)\theta^{\mathrm{T}}\theta - \theta_{\max}^2}{\epsilon_\theta \theta_{\max}^2}, \tag{2}$$

where $\theta_{\max} \in \mathbb{R}$ is a projection norm bound imposed on $\theta \in \mathbb{R}^n$ and $\epsilon_\theta > 0$ is a projection tolerance bound. Then, projection operator Proj:$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathrm{Proj}(\theta, y) \triangleq \begin{cases} y, & \text{if } \phi(\theta) < 0 \\ y, & \text{if } \phi(\theta) \geq 0 \\ & \text{and } \phi'(\theta)y \leq 0 \\ y - \frac{\phi'^{\mathrm{T}}(\theta)\phi'(\theta)y}{\phi'(\theta)\phi'^{\mathrm{T}}(\theta)}\phi(\theta), & \text{if } \phi(\theta) \geq 0 \\ & \text{and } \phi'(\theta)y > 0, \end{cases} \tag{3}$$

where $y \in \mathbb{R}^n$.

Remark 1. It follows from the definition projection operator that

$$(\theta - \theta^*)^{\mathrm{T}} \left(\mathrm{Proj}\left(\theta, y\right) - y\right) \leq 0, \quad \theta^* \in \mathbb{R}^n, \tag{4}$$

holds [10]. The definition of the projection operator can be generalized to matrices as

$$\begin{aligned} \mathrm{Proj_m}(\Theta, Y) \; = \; & \big(\mathrm{Proj}(\mathrm{col}_1(\Theta), \mathrm{col}_1(Y)), \ldots, \\ & \mathrm{Proj}(\mathrm{col}_m(\Theta), \mathrm{col}_m(Y))\big), \end{aligned} \tag{5}$$

where $\Theta \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{n\times m}$ and $\mathrm{col}_i(\cdot)$ denotes ith column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n\times m}$, it follows from (4) that

$$\begin{aligned} \mathrm{tr}\, \Big[(\Theta - \Theta^*)^{\mathrm{T}}(\mathrm{Proj_m}(\Theta, Y) - Y)\Big] = \sum_{i=1}^{m} \Big[\mathrm{col}_i(\Theta - \Theta^*)^{\mathrm{T}} \\ \cdot(\mathrm{Proj}(\mathrm{col}_i(\Theta), \mathrm{col}_i(Y)) - \mathrm{col}_i(Y))\Big] \leq 0. \end{aligned} \tag{6}$$

Next, consider the uncertain dynamical system subject to unmodeled dynamics given by

$$\begin{aligned} \dot{x}(t) \; &= \; Ax(t) + B\Lambda u(t) + B\delta(x(t)) + Bp(t), \\ & \qquad\qquad\qquad\qquad\qquad x(0) = x_0, \end{aligned} \tag{7}$$

$$\dot{q}(t) \; = \; Fq(t) + G\Lambda u(t), \quad q(0) = q_0, \tag{8}$$

$$p(t) \; = \; Hq(t), \tag{9}$$

where $x(t) \in \mathbb{R}^n$ is the measurable state vector, $u(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, $q(t) \in \mathbb{R}^p$ and $p(t) \in \mathbb{R}^m$ are respectively the unmodeled dynamics state and output vectors, $A \in \mathbb{R}^{n\times n}$ is known system matrix, $B \in \mathbb{R}^{n\times m}$ is known input matrix such that the pair (A, B) is controllable, $\Lambda \in \mathbb{R}_+^{m\times m} \cap \mathbb{D}^{m\times m}$ is unknown control effectiveness matrix, $\delta : \mathbb{R}^n \to R^m$ is a system uncertainty, and $F \in \mathbb{R}^{p\times p}$, $G \in \mathbb{R}^{p\times m}$, and $H \in \mathbb{R}^{m\times p}$ are matrices associated with unmodeled dynamics such that F is Hurwitz. Note that since $\Lambda \in \mathbb{R}_+^{m\times m} \cap \mathbb{D}^{m\times m}$, it follows that there exists $\lambda_L \in \mathbb{R}_+$ and $\lambda_U \in \mathbb{R}_+$ such that

$$\lambda_L \leq \|\Lambda\|_2 \leq \lambda_U, \tag{10}$$

holds. In addition, note that since F is Hurwitz, there exists $S \in \mathbb{R}_+^{p\times p} \cap \mathbb{S}^{p\times p}$ such that

$$0 = F^{\mathrm{T}}S + SF + I \tag{11}$$

holds.

Remark 2. Let

$$z(t) = \beta q(t), \quad z(t) \in \mathbb{R}^p, \tag{12}$$

where $\beta \in \mathbb{R}_+$ is a free variable to be used later in this paper. Then, the unmodeled dynamics given by (8) and (9) can be equivalently represented as

$$\dot{z}(t) \; = \; Fz(t) + \beta G\Lambda u(t), \quad z(0) = \beta q_0, \tag{13}$$

$$p(t) \; = \; \beta^{-1}Hq(t). \tag{14}$$

Using the state transformation given in Remark 2, the uncertain dynamical system subject to unmodeled dynamics can be equivalently written as

$$\begin{aligned} \dot{x}(t) \; &= \; Ax(t) + B\Lambda u(t) + B\delta\left(x(t)\right) \\ & \quad + \beta^{-1}BHz(t), \quad x(0) = x_0, \end{aligned} \tag{15}$$

$$\dot{z}(t) \; = \; Fz(t) + \beta G\Lambda u(t), \quad z(0) = \beta q_0. \tag{16}$$

Assumption 1. The system uncertainty $\delta : \mathbb{R}^n \to R^m$ can be parameterized as

$$\delta(x) = W_0^{\mathrm{T}}\sigma_0(x), \quad x \in \mathbb{R}^n, \tag{17}$$

where $W_0 \in \mathbb{R}^{s\times m}$ is an unknown weight matrix and $\sigma_0 \in \mathbb{R}^n \to \mathbb{R}^s$ is a known basis function on the form

$$\sigma_0\left(x\right) = \left[\sigma_{0_1}\left(x\right), \sigma_{0_2}\left(x\right), ..., \sigma_{0_s}\left(x\right)\right]^{\mathrm{T}}. \tag{18}$$

In addition, the basis function satisfies the inequality of the form given by

$$\|\sigma_0(x(t))\|_2 \quad \leq \quad l_0\|x(t)\|_2 + l_c, \quad x(t) \in \mathbb{R}^n, \quad (19)$$

where $l_0 \in \mathbb{R}_+$ and $l_c \in \mathbb{R}_+$.

Now, consider the reference system capturing a desired, closed-loop dynamical system performance given by

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}}x_{\mathrm{r}}(t) + B_{\mathrm{r}}c(t), \quad x_{\mathrm{r}}(0) = x_{r0}, \quad (20)$$

where $x_{\mathrm{r}}(t) \in \mathbb{R}^n$ is the reference state vector, $c(t) \in \mathbb{R}^m$ is a given uniformly continuous bounded command, $A_{\mathrm{r}} \in \mathbb{R}^{n\times n}$ is the Hurwitz reference system matrix, and $B_{\mathrm{r}} \in \mathbb{R}^{n\times m}$ is the command input matrix. Note that since A_{r} is Hurwitz, there exists $P \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ such that

$$0 = A_{\mathrm{r}}^{\mathrm{T}}P + PA_{\mathrm{r}} + I \quad (21)$$

holds. We now introduce the following standard assumption, which is known as the matching condition in the model reference adaptive control literature [3], [11], [12].

Assumption 2. There exist $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$ such that

$$A_{\mathrm{r}} \quad \triangleq \quad A - BK_1, \quad (22)$$
$$B_{\mathrm{r}} \quad \triangleq \quad BK_2 \quad (23)$$

hold.

Next, it follows from Assumptions 1 and 2 that (15) can be rewritten as

$$\dot{x}(t) \quad = \quad A_{\mathrm{r}}x(t) + B_{\mathrm{r}}c(t) + B\Lambda\left[u(t) + W^{\mathrm{T}}\sigma(\cdot)\right] + \beta^{-1}BHz(t), \quad (24)$$

where

$$W \quad \triangleq \quad \left[\Lambda^{-1}W_0^{\mathrm{T}}, \Lambda^{-1}K_1, -\Lambda^{-1}K_2\right]^{\mathrm{T}} \in \mathbb{R}^{q\times m}, \quad (25)$$
$$\sigma(\cdot) \quad \triangleq \quad \left[\sigma_0^{\mathrm{T}}(x(t)), x^{\mathrm{T}}(t), c^{\mathrm{T}}(t)\right]^{\mathrm{T}} \in \mathbb{R}^q, \quad (26)$$

with $q \triangleq s + n + m$. In addition, letting

$$e(t) \triangleq x(t) - x_{\mathrm{r}}(t), \quad (27)$$

be the system error, the system error dynamics can be given using (20) and (24) as

$$\dot{e}(t) \quad = \quad A_{\mathrm{r}}e(t) + B\Lambda\left[u(t) + W^{\mathrm{T}}\sigma(\cdot)\right] + \beta^{-1}BHz(t), \quad e(0) = e_0. \quad (28)$$

Considering (28), let the adaptive control law be

$$u(t) = -\hat{W}^{\mathrm{T}}(t)\sigma(\cdot) \quad (29)$$

where the $\hat{W}(t) \in \mathbb{R}^{q\times m}$ is an estimate of the unknown weight W satisfying the projection operator-based weight update law

$$\dot{\hat{W}}(t) \quad = \quad \gamma\mathrm{Proj_m}\left[\hat{W}(t),\ \sigma(\cdot)e^{\mathrm{T}}(t)PB\right], \quad \hat{W}(0) = \hat{W}_0 \geq 0, \quad (30)$$

with $\gamma \in \mathbb{R}_+$ being the learning rate. Note that since we utilize a projection bound in (30), it follows that

$$\|\hat{W}(t)\|_2 \leq w^*, \quad w^* \in \mathbb{R}_+. \quad (31)$$

Now using (29) respectively in (28) and (16), one can write

$$\dot{e}(t) \quad = \quad A_{\mathrm{r}}e(t) - B\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma(\cdot) + \beta^{-1}BHz(t), \quad e(0) = e0, \quad (32)$$
$$\dot{z}(t) \quad = \quad Fz(t) - \beta G\Lambda\hat{W}^{\mathrm{T}}(t)\sigma(\cdot), \quad z(0) = \beta q_0, \quad (33)$$

where

$$\tilde{W} \triangleq \hat{W}(t) - W \in \mathbb{R}^{q\times m}. \quad (34)$$

In the reminder of this section, we show boundedness of the closed-loop dynamical system — when a fundamental stability limit holds for the highlighted standard model reference adaptive control problem formulation.

Theorem 1. Consider the uncertain dynamical system subject to unmodeled dynamics given by (15) and (16), the reference system given by (20), the adaptive control law given by (29) and (30), and assume that Assumption 1, Assumption 2, and

$$\mathcal{R} \quad \triangleq \quad \begin{bmatrix} 1 & \eta \\ \eta & \alpha \end{bmatrix} > 0 \quad (35)$$

hold, where $\alpha \in \mathbb{R}_+$,

$$\eta \triangleq -\beta^{-1}\|PB\|_2\|H\|_2 - \alpha\beta\|SG\|_2\lambda_U w^* l, \quad (36)$$

and

$$l \triangleq (1 + l_0). \quad (37)$$

Then, the solution $(e(t), z(t), \tilde{W}(t))$ of the closed-loop dynamical system is uniformly bounded for all initial conditions.

Proof. To show uniform boundedness of the solution $(e(t), z(t), \tilde{W}(t))$ for all initial conditions, consider the Lyapunov-like function given by

$$\mathcal{V}(e, \tilde{W}, z) \quad = \quad e^{\mathrm{T}}Pe + \gamma^{-1}\mathrm{tr}(\tilde{W}\Lambda^{\frac{1}{2}})^{\mathrm{T}}(\tilde{W}\Lambda^{\frac{1}{2}}) + \alpha z^{\mathrm{T}}Sz. \quad (38)$$

Note that $\mathcal{V}(0,0,0) = 0$ and $\mathcal{V}(e, z, \tilde{W}) > 0$ for all $(e, z, \tilde{W}) \neq (0,0,0)$. Differentiating (38) along the closed-loop dynamical system trajectories yields

$$\begin{aligned}
&\dot{\mathcal{V}}(e(t), \tilde{W}(t), z(t)) \\
&\quad \leq -\|e(t)\|_2^2 - \alpha\|z(t)\|_2^2 + 2\beta^{-1}e^{\mathrm{T}}(t)PBHz(t) \\
&\qquad -2\alpha\beta z^{\mathrm{T}}(t)SG\Lambda\hat{W}^{\mathrm{T}}(t)\sigma(\cdot) \\
&\quad \leq -\|e(t)\|_2^2 - \alpha\|z(t)\|_2^2 + 2\beta^{-1}\|e(t)\|_2\|PB\|_2 \\
&\qquad \cdot\|H\|_2\|z(t)\|_2 + 2\alpha\beta\|z(t)\|_2\|SG\|_2\|\Lambda\|_2 \\
&\qquad \cdot\left\|\hat{W}(t)\right\|_2\|\sigma(\cdot)\|_2. \quad (39)
\end{aligned}$$

Next, an upper bound for $\|\sigma(\cdot)\|_2$ can be given using

Assumption 1 as

$$\begin{aligned}\|\sigma(\cdot)\|_2 &\le \|\sigma_0(x(t))\|_2 + \|x(t)\|_2 + \|c(t)\|_2 \\ &\le l\,\|e(t)\|_2 + d^*, \end{aligned} \tag{40}$$

where $d^* \in \mathbb{R}_+$ is an upper bound for

$$l\,\|x_{\rm r}(t)\|_2 + \|c(t)\|_2 + l_c, \tag{41}$$

that is

$$l\,\|x_{\rm r}(t)\|_2 + \|c(t)\|_2 + l_c \le d^*, \tag{42}$$

since $x_{\rm r}(t)$ and $c(t)$ are bounded. Now, it follows from (39) and (40) that

$$\begin{aligned}&\dot{\mathcal{V}}(e(t), \tilde{W}(t), z(t)) \\ &\quad\le -\|e(t)\|_2^2 - \alpha\,\|z(t)\|_2^2 + 2\beta^{-1}\,\|e(t)\|_2\,\|z(t)\|_2 \\ &\qquad\cdot \|PB\|_2\,\|H\|_2 + 2\alpha\beta\,\|z(t)\|_2\,\|SG\|_2\,\lambda_U w^* \\ &\qquad\cdot (l\,\|e(t)\|_2 + d^*) \\ &\quad= -\xi^{\rm T}(t)\mathcal{R}\xi(t) + 2\alpha\beta\,\|z(t)\|_2\,\|SG\|_2\,\lambda_U w^* d^*, \end{aligned} \tag{43}$$

where

$$\xi(t) \triangleq \left[\|e(t)\|_2, \|z(t)\|_2\right]^{\rm T}. \tag{44}$$

Finally, (43) can be rewritten as

$$\begin{aligned}&\dot{\mathcal{V}}(e(t), \tilde{W}(t), z(t)) \\ &\quad\le -\underline{\lambda}(\mathcal{R})\,\|\xi(t)\|_2 \left[\|\xi(t)\|_2 - \frac{r_0}{\underline{\lambda}(\mathcal{R})}\right], \end{aligned} \tag{45}$$

where

$$r_0 \triangleq 2\alpha\beta\,\|SG\|_2\,\lambda_U w^* d^*, \tag{46}$$

and hence, there exists a compact set such that

$$\dot{\mathcal{V}}(e(t), z(t), \tilde{W}(t)) < 0, \tag{47}$$

outside of this set, which proves the uniform boundedness of the closed-loop solution $(e(t), z(t), \tilde{W}(t))$ for all initial conditions [11], [13]. □

An interpretation of the assumption (35) is given next. Specifically, it can be shown that there exists $\alpha \in \mathbb{R}_+$ such that (35) holds if

$$\|SG\|_2\,\|H\|_2 < \frac{1}{4w^* l \lambda_U\,\|PB\|_2}. \tag{48}$$

Hence, it follows that the solution $(e(t), z(t), \tilde{W}(t))$ of the closed-loop dynamical system is uniformly bounded for all initial conditions when the fundamental stability limit given by (48) holds. In particular, it can be readily seen from (48) that the closed-loop dynamical system remains bounded either if there does not exist significant unmodeled dynamics or the effect of system uncertainties is negligible. Hence, the standard model reference adaptive control formulation considered in this section cannot tolerate large system uncertainties even when unmodeled dynamics satisfy a set of conditions. A version of this fundamental stability limit is also revealed in [5].

III. Relaxing Fundamental Stability Limit of Model Reference Adaptive Controllers in the Presence of Unmodeled Dynamics

We now propose an adaptive robustifying term that augments (29) to relax the fundamental stability limit given by (48). Specifically, let the adaptive control law be

$$u(t) = -\hat{W}^{\rm T}(t)\sigma(\cdot) - \hat{\mu}(t)B^{\rm T}Pe(t), \tag{49}$$

where $\hat{W}(t)$ satisfies the weight update law given by (30) and $\hat{\mu}(t)$ is a projection operator-based adaptive robustifying term given by

$$\begin{aligned}\dot{\hat{\mu}}(t) &= \mu_0 \mathrm{Proj}\left(\hat{\mu}(t),\ \left\|B^{\rm T}Pe(t)\right\|_2^2 - \sigma_\mu \hat{\mu}(t)\right), \\ &\qquad\qquad \hat{\mu}(0) = \hat{\mu}_0, \quad \hat{\mu}(0) \in \overline{R}_+, \end{aligned} \tag{50}$$

with $\mu_0 \in \mathbb{R}_+$ and $\sigma_\mu \in \mathbb{R}_+$ being design parameters. Note that since $\hat{\mu}(0) \in \overline{R}_+$, $\hat{\mu}(t) \in \overline{R}_+$ holds. In addition, we select the projection bound for (50) as

$$\hat{\mu}(t) \le \mu\psi, \quad \psi > 0, \tag{51}$$

$$\mu \triangleq \frac{2\lambda_L {\lambda_U}^2 l^2 {w^*}^2}{\epsilon - {\lambda_U}^2\psi^2}, \tag{52}$$

where $\epsilon \in \mathbb{R}_+$ is chosen such that

$$\epsilon > {\lambda_U}^2\psi^2, \tag{53}$$

holds.

Remark 3. The results in this section also holds without the leakage term (i.e., $-\sigma_\mu\hat{\mu}(t)$) in (50); however, we utilize this term to drive $\hat{\mu}(t)$ closer to zero for instants when the effect of $\left\|B^{\rm T}Pe(t)\right\|_2^2$ in (50) becomes small.

Next, using (49) respectively in (28) and (16), one can write

$$\begin{aligned}\dot{e}(t) &= A_{\rm r}e(t) - B\Lambda\tilde{W}^{\rm T}(t)\sigma(\cdot) - \hat{\mu}(t)B\Lambda B^{\rm T}Pe(t) \\ &\quad + \beta^{-1}BHz(t), \quad e(0) = e_0, \end{aligned} \tag{54}$$

$$\begin{aligned}\dot{z}(t) &= Fz(t) - \beta G\Lambda\hat{W}^{\rm T}(t)\sigma(\cdot) \\ &\quad - \beta\hat{\mu}(t)G\Lambda B^{\rm T}Pe(t), \quad z(0) = z_0. \end{aligned} \tag{55}$$

In the reminder of this section, we show boundedness of the closed-loop dynamical system with the proposed adaptive robustifying term and show that the proposed approach relaxes the fundamental stability limit given by (48).

Theorem 2. Consider the uncertain dynamical system subject to unmodeled dynamics given by (15) and (16), the reference system given by (20), the adaptive control law given by (49), (30), and (50), and assume that Assumption 1, Assumption 2, and

$$\mathcal{R} \triangleq \begin{bmatrix} 1 & 0 & \eta_1 \\ 0 & 2\mu\lambda_L & \eta_2 \\ \eta_1 & \eta_2 & \alpha \end{bmatrix} > 0, \tag{56}$$

hold, where $\alpha \in \mathbb{R}_+$,

$$\eta_1 \triangleq -\alpha\beta\lambda_U w^*\,\|SG\|_2\,l, \tag{57}$$

and

$$\eta_2 \triangleq -\beta^{-1} \|H\|_2 - \alpha\beta\mu\psi\lambda_U \|SG\|_2 . \quad (58)$$

Then, the solution $(e(t), \tilde{W}(t), z(t), \tilde{\mu}(t))$ of the closed-loop dynamical system is uniformly bounded for all initial conditions, where

$$\tilde{\mu}(t) \triangleq \hat{\mu}(t) - \mu. \quad (59)$$

Proof. The result follows from the consideration of the Lyapunov-like function given by

$$\begin{aligned} \mathcal{V}(e, \tilde{W}, z, \tilde{\mu}) &= e^{\mathrm{T}} P e + \gamma^{-1} \mathrm{tr}(\tilde{W}\Lambda^{\frac{1}{2}})^{\mathrm{T}} (\tilde{W}\Lambda^{\frac{1}{2}}) \\ &\quad + \alpha z^{\mathrm{T}} S z + {\mu_0}^{-1} \tilde{\mu}^2 \lambda_L, \end{aligned} \quad (60)$$

where $\mathcal{V}(0,0,0,0) = 0$ and $\mathcal{V}(e, \tilde{W}, z, \tilde{\mu}) > 0$ for all $(e, \tilde{W}, z, \tilde{\mu}) \neq (0,0,0,0)$, grouping the terms in the time-derivative of (60) with respect to $\|e(t)\|_2$, $\|B^{\mathrm{T}} P e(t)\|_2$, and $\|z(t)\|_2$ that yields (56), and utilizing the steps highlighted in the proof of Theorem 1. □

Similar to Section II, an interpretation of the assumption (56) is given next, where

$$\lambda_L > \frac{1}{2} \quad (61)$$

is assumed. Specifically, it can be shown that if

$$\|SG\|_2 \|H\|_2 < \frac{-\lambda_U \psi + \sqrt{\lambda_U^2 \psi^2 + \epsilon(2\lambda_L - 1)}}{\epsilon}, \quad (62)$$

holds, then (56) is satisfied. Unlike the fundamental stability limit given by (48) for the standard model reference adaptive control problem, it is of practical importance to note that the new fundamental stability limit given by (62) does not depend on upper bounds of the system uncertainty, and hence, it is relaxed in this sense. That is, the proposed framework allows the closed-loop dynamical system to remain bounded in the presence of large system uncertainties when the unmodeled dynamics satisfy the condition given by (62).

IV. Illustrative Numerical Example

Consider a second-order dynamical system subject to nonminimum phase unmodeled dynamics given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ \kappa_1 & \kappa_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Lambda u(t) \\ &\quad + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -3.5503 \\ -1.2836 \end{bmatrix}^{\mathrm{T}} z(t), \quad x(0) = 0, \quad (63) \\ \dot{z}(t) &= \begin{bmatrix} -0.05 & 3.1619 \\ -3.1619 & -0.05 \end{bmatrix} z(t) \\ &\quad + \begin{bmatrix} 0.0084 \\ 0.0002 \end{bmatrix} \Lambda u(t), \quad z(0) = 0, \quad (64) \end{aligned}$$

where κ_1, κ_2, and Λ are unknown parameters. Note that (63) and (64) can be equivalently represented in the form given by (15) and (16). For our illustrative example, we set $\kappa_1 = 1$, $\kappa_2 = 1$, and $\Lambda = 0.975$. Both for the results in Sections II and III, we select a reference system subject to zero initial conditions with natural frequency of $w_n = 0.3$rad/s and a damping ratio $r_n = 0.9$ such that

$$A_{\mathrm{r}} = \begin{bmatrix} 0 & 1 \\ -w_n^2 & -2w_n r_n \end{bmatrix}, \ B_{\mathrm{r}} = \begin{bmatrix} 0 \\ w_n^2 \end{bmatrix}, \quad (65)$$

hold.

Figures 1 and 2 show the closed-loop dynamical system performance with the standard model reference adaptive control approach in Theorem 1 with $\gamma = 0.6$ and the projection bound $\hat{W}_{\max} = 2$. Although the system performance presented in these figures remains bounded, the fundamental stability limit given by (48) does not hold (to compute this limit, we use $\lambda_L = 0.95$, $\lambda_U = 1$, $l = 1$, and $w^* = 3.4641$, which results in

$$\|SG\|_2 \|H\|_2 = 0.3161, \quad (66)$$

and

$$\frac{1}{4w^* l \lambda_U \|PB\|_2} = 0.0058, \quad (67)$$

for (48)). Thus, there does not exist any guarantees in general on the boundedness of the closed-loop signals.

Figures 3 and 4 show the closed-loop dynamical system performance with the proposed approach in Theorem 2 with $\gamma = 0.6$, $\mu_0 = 1.1$, $\sigma_\mu = 0.0001$, and the projection bounds $\hat{\mu}_{\max} = 2$ and $\hat{W}_{\max} = 2$. In this case, the fundamental stability limit given by (62) holds (to compute this limit, we use $\lambda_L = 0.95$, $\lambda_U = 1$, $\psi = 1.05$, and $\epsilon = 1.2\lambda_U^2$, which results in

$$\|SG\|_2 \|H\|_2 = 0.3161, \quad (68)$$

and

$$\frac{-\lambda_U \psi + \sqrt{\lambda_U^2 \psi^2 + \epsilon(2\lambda_L - 1)}}{\epsilon} = 0.3561, \quad (69)$$

for (62)). Thus, the proposed approach rigorously guarantees the boundedness of the closed-loop signals in the presence of considered class of unmodeled dynamics.

V. Conclusion

It is well known that there exists a fundamental stability limit for standard model reference adaptive controllers in the presence of unmodeled dynamics such that they cannot guarantee closed-loop dynamical system stability in the face of large system uncertainty levels. Motivated from this viewpoint, we presented a projection operator-based adaptive robustifying term that augments a projection operator-based adaptive control signal. Specifically, the proposed framework allows the closed-loop dynamical system to remain stable in the presence of large system uncertainties when the unmodeled dynamics satisfy a set of conditions without

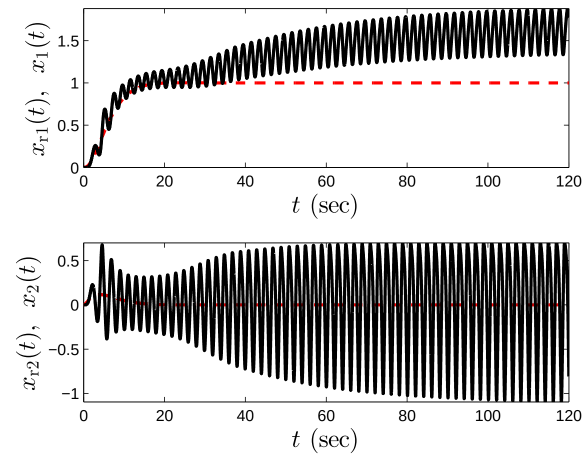

Fig. 1. System state and reference state responses for the standard model reference adaptive controller in Theorem 1.

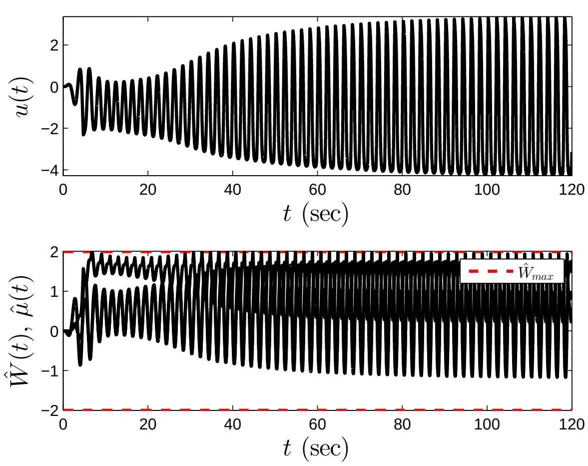

Fig. 2. Control signal and weight update law responses for the standard model reference adaptive controller in Theorem 1.

making assumptions on initial conditions and persistency of excitation. Illustrative numerical examples demonstrated the efficacy of the proposed approach. Finally, it should be noted that the proofs yielding (48) and (62) as well as the details on the proof of Theorem 2 is omitted from this paper due to page limitations and will be reported elsewhere.

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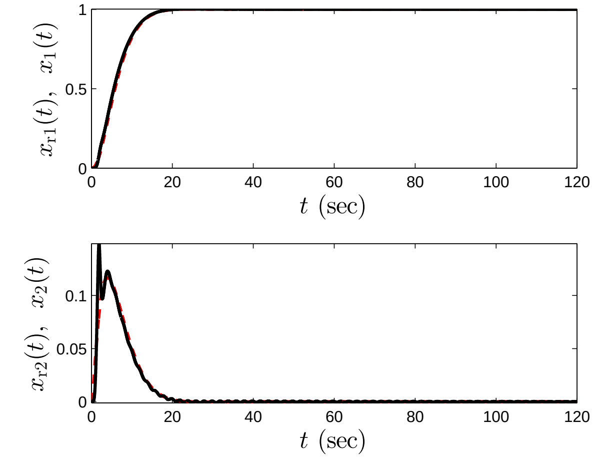

Fig. 3. System state and reference state responses for the proposed model reference adaptive controller in Theorem 2.

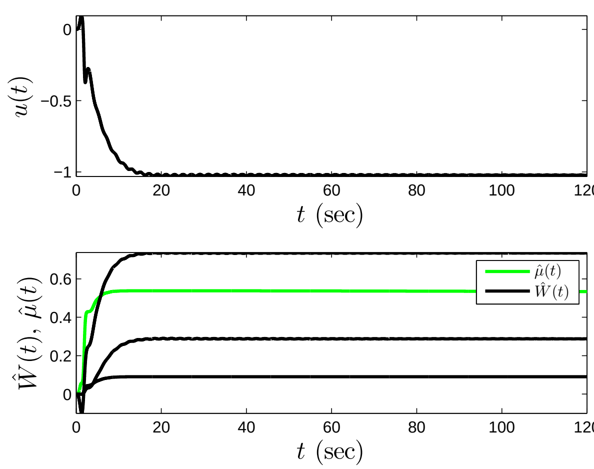

Fig. 4. Control signal and weight update law responses for the proposed model reference adaptive controller in Theorem 2.

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