Frequency-Limited Adaptive Control Architecture for Transient Response Improvement

Tansel Yucelen, Gerardo De La Torre, and Eric N. Johnson

Abstract— **This paper presents a new adaptive control architecture to achieve stabilization and command following of uncertain dynamical systems with improved transient performance. Our framework consists of a new reference system and an adaptive controller. The proposed reference system captures a desired closed-loop dynamical system behavior modified by a mismatch term representing the high-frequency content between the uncertain dynamical system and this reference system, i.e., the system error. In particular, this mismatch term allows one to limit the frequency content of the system error dynamics, which is used to drive the adaptive controller. It is shown that this key feature of our framework yields fast adaptation without incurring high-frequency oscillations in the transient performance.**

I. Introduction

The control framework of this paper builds on a well-known and important class of adaptive controllers, specifically, model reference adaptive controllers. Whitaker *et al.* [1, 2] originally proposed the model reference adaptive control concept. In particular, model reference adaptive control schemes have three major components, namely, a reference system (model), an update law, and a controller. The reference system, in the classical sense, captures a desired closed-loop dynamical system behavior for which its output (resp., state) is compared with the output (resp., state) of the uncertain dynamical system. This comparison results in a system error signal used to drive the update law online. Then, the controller adapts feedback gains to minimize this error signal using the information received from the update law. From a practical standpoint, it should be noted that the output (resp., state) of the uncertain dynamical system can be far different from the output (resp., state) of the reference system during transient time (learning phase), although a model reference adaptive control scheme can guarantee that the distance between the uncertain dynamical system and the reference system vanishes asymptotically. This problem, so-called poor transient performance phenomenon, can be solved by increasing the learning rate of the update law, and hence, fast adaptation can be achieved in order to suppress uncertainties rapidly during transient time.

Update laws with high learning rates may yield to signals with high-frequency content, which can, for example, excite unmodeled system dynamics [3] resulting in system instability for practical applications. Hence, a critical trade-off between system stability and control adaptation rate exists in most adaptive control approaches, with some notable exceptions [4–6]. The authors in [4] use a low-pass filter that subverts high-frequency oscillations attributable to fast adaptation, and their approach has guaranteed transient and steady-state performance. Even though this methodology is promising, it requires the knowledge of a conservative upper bound on the unknown constant gain appearing in their uncertainty parameterization. While this conservative upper bound can be available for some applications, the actual upper bound may change and exceed its conservative estimate, for example, when an aircraft in a flight control application undergoes a sudden change in dynamics. The author in [5] presents a modification to the reference system in order to solve the poor transient performance phenomenon, where a detailed analysis of this approach is given in [6]. Specifically, this modification is constructed by using a modification gain multiplied by the system error that is between the uncertain dynamical system and the modified reference system. In the limit as this modification gain goes to infinity, it is shown that the system error goes to zero in transient time. This approach can be used to effectively suppress uncertainties, however, for example, in the presence of exogenous low-frequency persistent disturbances, the transient performance of this approach may not be sufficient. Because, this disturbance may not be *visible* to the update law, since the system error is (sufficiently) small due to a (sufficiently) large modification gain.

This paper presents a new adaptive control architecture to achieve stabilization and command following of uncertain dynamical systems with improved transient performance. Our framework consists of a new reference system and an adaptive controller. The proposed reference system captures a desired closed-loop dynamical system behavior modified by a mismatch term representing the high-frequency content between the uncertain dynamical system and this reference system, i.e., the system error (Fig. 1). In particular, this mismatch term allows one to limit the frequency content of the system error dynamics, which is used to drive the adaptive controller. That is, the purpose of our methodology is to prevent the update law from attempting to learn through

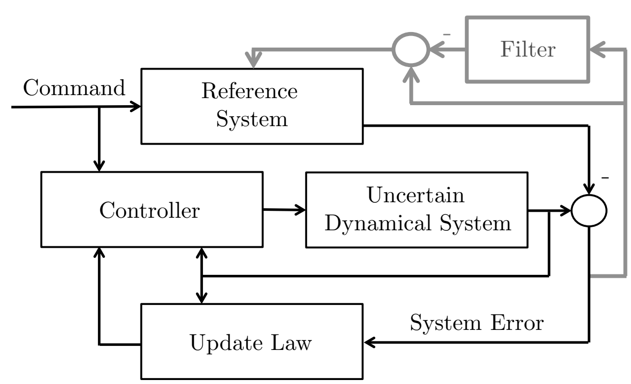

Fig. 1. Block diagram of the proposed scheme. Note that the reference system is driven not only by the command but also by the difference between the system error and its (low-pass) filtered form representing the high-frequency content of the system error.

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the high-frequency content of the system error, and hence, the update law only learns through the low-frequency content of the system error, which constitutes a distinction over the approach in [5]. It is shown that this key feature of our framework yields fast adaptation without incurring high-frequency oscillations in the transient performance.

On the notation used throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp., $\overline{\mathbb{R}}_+$) denotes the set of positive (resp., nonnegative-definite) real numbers, $\mathbb{R}_+^{n\times n}$ (resp., $\overline{\mathbb{R}}_+^{n\times n}$) denotes the set of $n \times n$ positive-definite (resp., nonnegative-definite) real matrices, $\mathbb{S}^{n\times n}$ denotes the set of $n \times n$ symmetric real matrices, $\mathbb{D}^{n\times n}$ denotes the $n \times n$ real matrices with diagonal scalar entries, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, and "$\triangleq$" denotes equality by definition. In addition, we write $\lambda_{\min}(A)$ (resp., $\lambda_{\max}(A)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix A, $\mathrm{tr}(\cdot)$ for the trace operator, $\mathrm{vec}(\cdot)$ for the column stacking operator, $\|\cdot\|_2$ for the Euclidian norm, $\|\cdot\|_\infty$ for the infinity norm, and $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm. Furthermore, for a signal

$$x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^n, \tag{1}$$

defined for all $t \geq 0$, the truncated $\mathcal{L}_\infty$ norm and the $\mathcal{L}_\infty$ norm [7, Section 5] are defined as

$$\|x_\tau(t)\|_{\mathcal{L}_\infty} \triangleq \max_{1\leq i\leq n} \big(\sup_{0\leq t\leq\tau} |x_i(t)|\big), \tag{2}$$

and

$$\|x(t)\|_{\mathcal{L}_\infty} \triangleq \max_{1\leq i\leq n} \big(\sup_{t\geq 0} |x_i(t)|\big), \tag{3}$$

respectively.

II. Preliminaries

We begin by presenting a brief review of the model reference adaptive control problem. Specifically, consider the nonlinear uncertain dynamical system given by

$$\dot{x}_{\mathrm{p}}(t) = A_{\mathrm{p}} x_{\mathrm{p}}(t) + B_{\mathrm{p}} \Lambda u(t) + B_{\mathrm{p}} \delta_{\mathrm{p}}\big(x_{\mathrm{p}}(t)\big), \quad x_{\mathrm{p}}(0) = x_{\mathrm{p}_0}, \tag{4}$$

where $x_{\mathrm{p}}(t) \in \mathbb{R}^{n_{\mathrm{p}}}$ is the accessible state vector, $u(t) \in \mathbb{R}^m$ is the control input, $\delta_{\mathrm{p}} : \mathbb{R}^{n_{\mathrm{p}}} \to \mathbb{R}^m$ is an *uncertainty*, $A_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}\times n_{\mathrm{p}}}$ is a known system matrix, $B_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}\times m}$ is a known control input matrix, and $\Lambda \in \mathbb{R}_+^{m\times m} \cap \mathbb{D}^{m\times m}$ is an *unknown* control effectiveness matrix. Furthermore, we assume that the pair $(A_{\mathrm{p}}, B_{\mathrm{p}})$ is controllable and the uncertainty is parameterized as

$$\delta_{\mathrm{p}}(x_{\mathrm{p}}) = W_{\mathrm{p}}^{\mathrm{T}} \sigma_{\mathrm{p}}(x_{\mathrm{p}}), \quad x_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{p}}}, \tag{5}$$

where $W_{\mathrm{p}} \in \mathbb{R}^{s\times m}$ is an *unknown* weight matrix and $\sigma_{\mathrm{p}} : \mathbb{R}^{n_{\mathrm{p}}} \to \mathbb{R}^s$ is a known basis function of the form

$$\sigma_{\mathrm{p}}(x_{\mathrm{p}}) = \big[\sigma_{\mathrm{p}_1}(x_{\mathrm{p}}), \sigma_{\mathrm{p}_2}(x_{\mathrm{p}}), \ldots, \sigma_{\mathrm{p}_s}(x_{\mathrm{p}})\big]^{\mathrm{T}}. \tag{6}$$

To address command following, let $c(t) \in \mathbb{R}^{n_{\mathrm{c}}}$ be a given bounded piecewise continuous command and $x_{\mathrm{c}}(t) \in \mathbb{R}^{n_{\mathrm{c}}}$ be the integrator state satisfying

$$\dot{x}_{\mathrm{c}}(t) = E_{\mathrm{p}} x_{\mathrm{p}}(t) - c(t), \quad x_{\mathrm{c}}(0) = x_{\mathrm{c}_0}, \tag{7}$$

where $E_{\mathrm{p}} \in \mathbb{R}^{n_{\mathrm{c}}\times n_{\mathrm{p}}}$ allows to choose a subset of $x_{\mathrm{p}}(t)$ to be followed by $c(t)$. Now, (4) is augmented with (7) as

$$\dot{x}(t) = Ax(t) + B\Lambda u(t) + BW_{\mathrm{p}}^{\mathrm{T}} \sigma_{\mathrm{p}}\big(x_{\mathrm{p}}(t)\big) + B_{\mathrm{r}} c(t), \quad x(0) = x_0, \tag{8}$$

where

$$x(t) \triangleq \big[x_{\mathrm{p}}^{\mathrm{T}}(t), x_{\mathrm{c}}^{\mathrm{T}}(t)\big]^{\mathrm{T}} \in \mathbb{R}^n, \quad n = n_{\mathrm{p}} + n_{\mathrm{c}}, \tag{9}$$

is the (augmented) state vector, $x_0 \triangleq \big[x_{\mathrm{p}_0}^{\mathrm{T}}, x_{\mathrm{c}_0}^{\mathrm{T}}\big]^{\mathrm{T}} \in \mathbb{R}^n$,

$$A \triangleq \begin{bmatrix} A_{\mathrm{p}} & 0_{n_{\mathrm{p}}\times n_{\mathrm{c}}} \\ E_{\mathrm{p}} & 0_{n_{\mathrm{c}}\times n_{\mathrm{c}}} \end{bmatrix} \in \mathbb{R}^{n\times n}, \tag{10}$$

$B \triangleq \big[B_{\mathrm{p}}^{\mathrm{T}}, 0_{n_{\mathrm{c}}\times m}^{\mathrm{T}}\big]^{\mathrm{T}} \in \mathbb{R}^{n\times m}$, and $B_{\mathrm{r}} \triangleq \big[0_{n_{\mathrm{p}}\times n_{\mathrm{c}}}^{\mathrm{T}}, -I_{n_{\mathrm{c}}\times n_{\mathrm{c}}}\big]^{\mathrm{T}} \in \mathbb{R}^{n\times n_{\mathrm{c}}}$.

Next, consider the feedback control law given by

$$u(t) = u_{\mathrm{n}}(t) + u_{\mathrm{a}}(t), \tag{11}$$

where $u_{\mathrm{n}}(t) \in \mathbb{R}^m$ and $u_{\mathrm{a}}(t) \in \mathbb{R}^m$ are the nominal and adaptive control laws, respectively. Furthermore, let the nominal control law be

$$u_{\mathrm{n}}(t) = -Kx(t), \quad K \in \mathbb{R}^{m\times n}, \tag{12}$$

such that $A_{\mathrm{r}} \triangleq A - BK$ is Hurwitz. Using (11) and (12) in (8) yields

$$\dot{x}(t) = A_{\mathrm{r}} x(t) + B_{\mathrm{r}} c(t) + B\Lambda\big[u_{\mathrm{a}}(t) + W^{\mathrm{T}} \sigma\big(x(t)\big)\big], \tag{13}$$

where

$$W^{\mathrm{T}} \triangleq \big[\Lambda^{-1} W_{\mathrm{p}}^{\mathrm{T}}, (\Lambda^{-1} - I_{m\times m})K\big] \in \mathbb{R}^{(s+n)\times m}, \tag{14}$$

is an *unknown* (aggregated) weight matrix and

$$\sigma^{\mathrm{T}}\big(x(t)\big) \triangleq \big[\sigma_{\mathrm{p}}^{\mathrm{T}}\big(x_{\mathrm{p}}(t)\big), x^{\mathrm{T}}(t)\big] \in \mathbb{R}^{s+n}, \tag{15}$$

is a known (aggregated) basis function. Considering (13), let the adaptive control law be

$$u_{\mathrm{a}}(t) = -\hat{W}^{\mathrm{T}}(t) \sigma\big(x(t)\big), \tag{16}$$

where $\hat{W}(t) \in \mathbb{R}^{(s+n)\times m}$ be the estimate of W satisfying the update law

$$\dot{\hat{W}}(t) = \gamma \sigma\big(x(t)\big) e^{\mathrm{T}}(t) PB, \quad \hat{W}(0) = \hat{W}_0, \tag{17}$$

where $\gamma \in \mathbb{R}_+$ is the learning rate, $e(t) \triangleq x(t) - x_{\mathrm{r}}(t)$ is the system error with $x_{\mathrm{r}}(t) \in \mathbb{R}^n$ being the reference state vector satisfying the reference system

$$\dot{x}_{\mathrm{r}}(t) = A_{\mathrm{r}} x_{\mathrm{r}}(t) + B_{\mathrm{r}} c(t), \quad x_{\mathrm{r}}(0) = x_{\mathrm{r}_0}, \tag{18}$$

and $P \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$ is a solution of the Lyapunov equation

$$0 = A_{\mathrm{r}}^{\mathrm{T}} P + PA_{\mathrm{r}} + R, \tag{19}$$

with $R \in \mathbb{R}_+^{n\times n} \cap \mathbb{S}^{n\times n}$.

Now, the system error dynamics is given by using (13), (16), and (18) as

$$\dot{e}(t) = A_{\mathrm{r}} e(t) - B\Lambda \tilde{W}^{\mathrm{T}}(t) \sigma\big(x(t)\big), \quad e(0) = e_0, \tag{20}$$

where

$$\tilde{W}(t) \triangleq \hat{W}(t) - W \in \mathbb{R}^{(s+n)\times m}, \tag{21}$$

is the weight error and $e_0 \triangleq x_0 - x_{\mathrm{r}_0}$. The update law given by (17) can be derived by using Lyapunov analysis by considering the Lyapunov function candidate

$$\mathcal{V}\big(e,\tilde{W}\big) \;=\; e^{\mathrm{T}}Pe+\gamma^{-1}\mathrm{tr}\,\big(\tilde{W}\Lambda^{\frac{1}{2}}\big)^{\mathrm{T}}\big(\tilde{W}\Lambda^{\frac{1}{2}}\big). \quad (22)$$

Furthermore, differentiating (22) and then using Barbalat's lemma [7], it can be shown that the system error $e(t)$ and the weight error $\tilde{W}(t)$ are Lyapunov stable with

$$\lim_{t\to\infty} e(t)=0. \quad (23)$$

Remark 2.1. Although (23) holds, the state vector $x(t)$ can be far different from $x_{\mathrm{r}}(t)$ during transient time (learning phase), unless a high learning rate γ is used in the update law (17). As discussed, however, update laws with high learning rates in the face of large system uncertainties and abrupt changes may yield to signals with high-frequency oscillations.

III. Frequency-Limited Adaptive Control

One of the fundamental components of a model reference adaptive control scheme is the system error $e(t)$. In particular, if the system error $e(t)$ contains any high-frequency oscillations, then the adaptive control law (16) can also have such oscillations, since the update law (17) is driven by this system error $e(t)$. Motivating from this standpoint, our aim is to limit the frequency-content of the system error dynamics (20) during transient-time (learning phase), and hence, to filter out any possible high-frequency oscillations contained in the error signal $e(t)$.

A. Frequency-Limited System Error Dynamics

Let $e_{\mathrm{L}}(t)\in\mathbb{R}^n$ be a low-pass filtered system error of $e(t)$ given by

$$\dot{e}_{\mathrm{L}}(t) \;=\; A_{\mathrm{r}}e_{\mathrm{L}}(t)+\eta\big(e(t)-e_{\mathrm{L}}(t)\big), \quad e_{\mathrm{L}}(0)=0, \quad (24)$$

where $\eta\in\mathbb{R}_+$ is a filter gain. Note that since $e_{\mathrm{L}}(t)$ is a low-pass filtered system error of $e(t)$, the filter gain η is chosen such that $\eta\le\eta^*$, where $\eta^*\in\mathbb{R}_+$ is a design parameter. We add a mismatch term to the system error dynamics (20) in order to enforce a distance condition between the trajectories of the system error $e(t)$ and the trajectories of its low-pass filtered version $e_{\mathrm{L}}(t)$. This leads to a minimization problem involving an error criterion capturing the distance between $e(t)$ and $e_{\mathrm{L}}(t)$. In particular, consider the cost function given by

$$\mathcal{J}\big(e,e_{\mathrm{L}}\big)=\frac{1}{2}\|e-e_{\mathrm{L}}\|_2^2, \quad (25)$$

and note that the negative gradient of this cost function with respect to e is given by $-\big(e(t)-e_{\mathrm{L}}(t)\big)$, which gives the structure of the proposed mismatch term. Using the idea presented in [2, 8], we now need to add $-\big(e(t)-e_{\mathrm{L}}(t)\big)$ to the system error dynamics given by (20). For this purpose, we modify the reference system (18) as

$$\dot{x}_{\mathrm{r}}(t) \;=\; A_{\mathrm{r}}x_{\mathrm{r}}(t)+B_{\mathrm{r}}c(t)+\kappa\big(e(t)-e_{\mathrm{L}}(t)\big), \quad x_{\mathrm{r}}(0)=x_{\mathrm{r}_0}, \quad (26)$$

where $\kappa\in\mathbb{R}_+$, and hence, the system error dynamics is given by using (13), (16), and (26) as

$$\dot{e}(t) \;=\; A_{\mathrm{r}}e(t)-B\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma\big(x(t)\big)-\kappa\big(e(t)-e_{\mathrm{L}}(t)\big), \quad e(0)=e_0. \quad (27)$$

Finally, note for the rest of this paper that the update law (17) is driven by the system error $e(t)=x(t)-x_{\mathrm{r}}(t)$, where $x_{\mathrm{r}}(t)$ is obtained from (26) (not (18)).

Remark 3.1. The reference system (26) captures a desired closed-loop dynamical system behavior modified by a mismatch term $\kappa\big(e(t)-e_{\mathrm{L}}(t)\big)$ representing the high-frequency content between the uncertain dynamical system and this reference system. Although this implies a modification of the ideal (unmodified) reference system (18) during transient time, as we see in the following sections, this mismatch term allows to limit the frequency content of the system error dynamics (27), which is used to drive the adaptive controller. In other words, the purpose of our methodology is to prevent the update law from attempting to learn through the high-frequency content of the system error.

Remark 3.2. As it is noted, the filter gain η needs to be chosen such that $\eta\le\eta^*$, where η^* needs to be small enough to cut off the high-frequency content of $e(t)$. To see the negative effect of high filter gain, let η be sufficiently large. Then,

$$e(t)-e_{\mathrm{L}}(t)\approx 0 \quad (28)$$

as a consequence of (24), and hence, we approximately recover the ideal (unmodified) reference system given by (18). In this case, the proposed approach converges to a standard model reference adaptive control scheme, which has practical limitations as discussed earlier in the presence of high learning rate γ. Furthermore, as a special case of $\eta=0$, the proposed approach converges to the approach documented in [5], since $e_{\mathrm{L}}(t)\equiv 0$ for all $t\in\overline{\mathbb{R}}_+$ as a consequence of (24). Once again, as discussed earlier, this selection for the filter gain η may result in poor transient performance in the presence of exogenous low-frequency persistent disturbances. Therefore, from a practical point of view, this imposes another constraint in the selection of filter gain such that it also needs to satisfy $\eta_*\le\eta$, where $\eta_*\in\mathbb{R}_+$ needs to be large enough in order to suppress the effects of exogenous low-frequency persistent disturbances. This phenomenon is illustrated in the next subsection.

B. A Simple Illustrative Example

Before going any further, this subsection elucidates the mechanism behind our proposed approach. For this purpose, let $n_{\mathrm{p}}=1$, $n_{\mathrm{c}}=0$, $m=1$, $A_{\mathrm{p}}=-\alpha$, $\alpha\in\mathbb{R}_+$, $B_{\mathrm{p}}=\alpha$, $K=0$, and $\delta_{\mathrm{p}}\big(x_{\mathrm{p}}\big)=d$ with d denoting an exogenous low-frequency disturbance. Furthermore, set $R=2$ in (19) such that $P=\alpha^{-1}$ and let all initial conditions be zero. For this special case, the system loop transfer function $\mathcal{G}(s)$ (broken at the control input) can be equivalently written as a linear time-invariant dynamical system, and hence, we can resort to classical control theory tools, such as Bode plots, to analyze the closed-loop system with respect to different choices of γ, κ, and η. Specifically, the system loop transfer function is given by

$$\mathcal{G}(s) \;=\; \underbrace{\frac{\gamma}{s}\left(\frac{s+\alpha+\eta}{s+\alpha+\kappa+\eta}\right)}_{\mathcal{C}(s)}\underbrace{\left(\frac{\alpha}{s+\alpha}\right)}_{\mathcal{P}(s)}, \quad (29)$$

where $\mathcal{C}(s)$ and $\mathcal{P}(s)$ denote the controller and the plant, respectively. Furthermore, for the standard model reference adaptive controller (η is sufficiently large, or simply, $\kappa=0$), note that

$$\mathcal{C}(s)=\frac{\gamma}{s}. \quad (30)$$

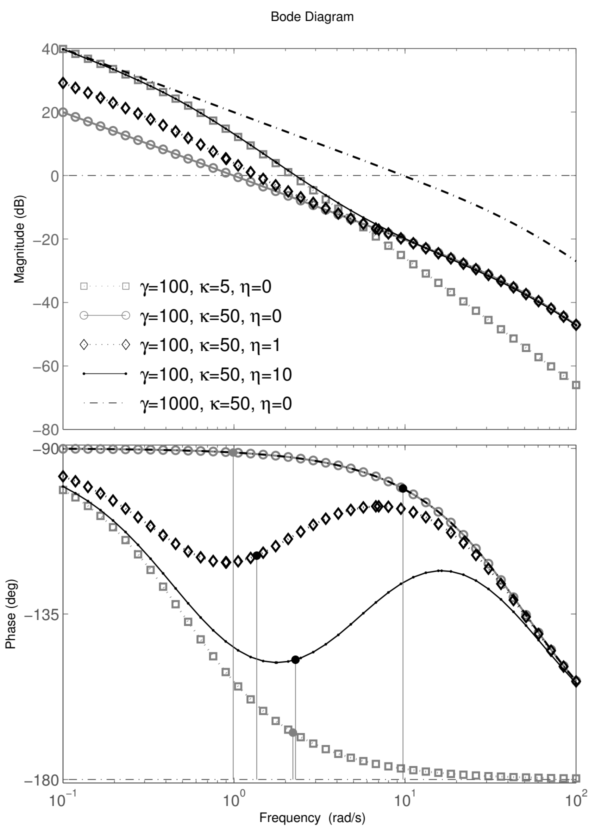

Fig. 2. Bode plots of the loop gain transfer function for different γ, κ, and η.

For the controller $\mathcal{C}(s)$ of proposed approach, since $\frac{\gamma}{s}$ is multiplied by a lead compensator

$$\frac{s+\alpha+\eta}{s+\alpha+\kappa+\eta}, \tag{31}$$

it can improve stability margins of the closed-loop system positively.

To further see the effects of γ, κ, and η, consider the Bode plot of $\mathcal{G}(s)$ in Fig. 2. Here, we tune these design parameters in order to obtain a large loop gain at low frequencies (from 0 rad/s to $5/2\pi$ rad/s) for good rejection of low-frequency disturbances and a small loop gain at high frequencies to avoid injecting too much measurement noise into the plant [9, Section 11.4]. Furthermore, we also would like to have at least a time-delay margin of 0.25 seconds. From Fig. 2, one can see that the cases $(\gamma,\kappa,\eta)=(100,5,0)$, $(\gamma,\kappa,\eta)=(100,50,10)$, and $(\gamma,\kappa,\eta)=(1000,50,0)$ achieves approximately the same rejection of low-frequency disturbances. However, it should be noted that the case $(\gamma,\kappa,\eta)=(1000,50,0)$ amplifies measurement noise excessively in comparison to the cases $(\gamma,\kappa,\eta)=(100,5,0)$ and $(\gamma,\kappa,\eta)=(100,50,10)$. In addition, it should be also noted that the case $(\gamma,\kappa,\eta)=(100,5,0)$ has the poorest time-delay margin of 0.1 seconds, whereas the cases $(\gamma,\kappa,\eta)=(100,50,10)$ and $(\gamma,\kappa,\eta)=(1000,50,0)$ have time-delay margins of 0.25 and 0.14 seconds, respectively. Therefore, one can conclude that the case $(\gamma,\kappa,\eta)=(100,50,10)$ achieves good rejection of low-frequency disturbances like the other two cases, has the maximum time-delay margin, and does not inject measurement noise as compared to the case $(\gamma,\kappa,\eta)=(1000,50,0)$. This shows the significance of having additional design parameters η and κ in the control design process. Moreover, the effect of increasing κ alone can be depicted from the cases $(\gamma,\kappa,\eta)=(100,5,0)$ and $(\gamma,\kappa,\eta)=(100,50,0)$. Specifically, it deteriorates the rejection properties of low-frequency disturbances. That is the reason why we increased the adaptation gain γ in the case $(\gamma,\kappa,\eta)=(1000,50,0)$ for achieving the same level of low-frequency disturbance rejection characteristics, however, as noted, this amplifies the measurement noise excessively and has less time-delay margin as compared to the case $(\gamma,\kappa,\eta)=(100,50,10)$.

Finally, the effect of increasing η to a moderate value can be seen from the cases $(\gamma,\kappa,\eta)=(100,50,0)$, $(\gamma,\kappa,\eta)=(100,50,1)$, and $(\gamma,\kappa,\eta)=(100,50,10)$. That is, we can recover the desired low-frequency disturbance rejection characteristics without increasing γ, and hence, without amplifying the measurement noise.

C. Transient and Steady-State Performance Guarantees

To establish transient and steady-state performance properties of the proposed adaptive control architecture, consider $e(t)=x(t)-x_{\mathrm{r}}(t)$ with $x_{\mathrm{r}}(t)$ satisfying (26) and $\tilde{W}(t)=\hat{W}(t)-W$. Furthermore, let the ideal (unmodified) reference system[1] be

$$\dot{x}_{\mathrm{r_i}}(t) \;=\; A_{\mathrm{r}}x_{\mathrm{r_i}}(t)+B_{\mathrm{r}}c(t), \quad x_{\mathrm{r_i}}(0)=x_{\mathrm{r_0}}, \tag{32}$$

where $x_{\mathrm{r_i}}(t)\in\mathbb{R}^n$ being the ideal reference state vector. Finally, let

$$\tilde{x}(t)\triangleq x_{\mathrm{r}}(t)-x_{\mathrm{r_i}}(t), \tag{33}$$

be the deviation error from the ideal reference system with $x_{\mathrm{r}}(t)$, once again, satisfying (26). Then, the system error, weight update error, low-pass filtered system error, and the deviation error dynamics are, respectively, given by (27), (24),

$$\dot{\tilde{W}}(t) \;=\; \gamma\sigma\big(x(t)\big)e^{\mathrm{T}}(t)PB, \quad \tilde{W}(0)=\tilde{W}_0, \tag{34}$$

$$\dot{\tilde{x}}(t) \;=\; A_{\mathrm{r}}\tilde{x}(t)+\kappa\big(e(t)-e_{\mathrm{L}}(t)\big), \quad \tilde{x}(0)=0, \tag{35}$$

where $\tilde{W}_0\triangleq\hat{W}_0-W$. The next theorem highlights transient and steady-state performance guarantees of our proposed approach.

Theorem 3.1. Consider the nonlinear uncertain dynamical system given by (4) subject to (5), the (modified) reference system given by (26), and the feedback control law given by (11) along with (12), (16), and (17). Then, the solution $\big(e(t),\tilde{W}(t),e_{\mathrm{L}}(t),\tilde{x}(t)\big)$ given by (27), (24), (34), and (35) is Lyapunov stable for all

$$\big(e_0,\tilde{W}_0,0,0\big)\in\mathbb{R}^n\times\mathbb{R}^{(s+n)\times m}\times\mathbb{R}^n\times\mathbb{R}^n, \tag{36}$$

and $t\in\overline{\mathbb{R}}_+$, and

$$\lim_{t\to\infty}\big(x(t)-x_{\mathrm{r_i}}(t)\big)=0. \tag{37}$$

For $t\in\overline{\mathbb{R}}_+$, in addition,

[1] To prevent any abuse of notation, we redefine the ideal (unmodified) reference system in (18) as (32).

$$\|x(t)-x_{\mathrm{r_i}}(t)\|_{\mathcal{L}_\infty} \leq \sqrt{\frac{\epsilon_{\mathcal{V}}}{\lambda_{\min}(P)}}\left(1+\sqrt{\frac{\kappa\lambda_{\max}(P)}{2\xi\lambda_{\min}(R)}}\right), \quad (38)$$

where

$$\xi \in (0,1) \quad (39)$$

and

$$\epsilon_{\mathcal{V}} \triangleq \gamma^{-1}\|\tilde{W}_0\Lambda^{\frac{1}{2}}\|_{\mathrm{F}}^2 + \lambda_{\max}(P)\|e_0\|_2^2. \quad (40)$$

Proof. Due to page limitation, the proof is omitted here. However, it follows by considering the Lyapunov function candidate

$$\mathcal{V}^*\big(e,\tilde{W},e_{\mathrm{L}},\tilde{x}\big) = \mathcal{V}\big(e,\tilde{W}\big)+\eta^{-1}\kappa e_{\mathrm{L}}^{\mathrm{T}}Pe_{\mathrm{L}} + 2\xi\kappa^{-1} \cdot\lambda_{\max}^{-1}(P)\lambda_{\min}(R)\tilde{x}^{\mathrm{T}}P\tilde{x}, \quad (41)$$

and then extending the analyzes in [4] and [10]. □

Theorem 3.1 highlights not only stability but also transient and steady-state performance guarantees of the nonlinear uncertain dynamical system subject to the proposed adaptive control architecture. Specifically, even though the proposed architecture is predicated on a modified reference system given by (26), Theorem 3.1 shows that

$$\lim_{t\to\infty}\big(x(t)-x_{\mathrm{r_i}}(t)\big)=0, \quad (42)$$

that is the (augmented) state vector $x(t)$ of (8) asymptotically converges to the ideal reference state vector $x_{\mathrm{r_i}}(t)$ of (32). Furthermore, during transient time (learning phase), the worst-case transient performance bound between $x(t)$ and $x_{\mathrm{r_i}}(t)$ is given by (38).

To further elucidate this performance bound, we let $x_{\mathrm{r_0}} = x_0$ in (26), and hence, $e(0) = 0$ in (27). Now, denoting

$$\epsilon_{\mathcal{V}_1} \triangleq \|\tilde{W}_0\Lambda^{\frac{1}{2}}\|_{\mathrm{F}}/\sqrt{\lambda_{\min}(P)}, \quad (43)$$

and

$$\epsilon_{\mathcal{V}_2} \triangleq \sqrt{\frac{1}{2}\xi^{-1}\lambda_{\min}^{-1}(R)\lambda_{\min}(P)}, \quad (44)$$

it follows from (38) that

$$\|x(t)-x_{\mathrm{r_i}}(t)\|_{\mathcal{L}_\infty} \leq \gamma^{-\frac{1}{2}}\epsilon_{\mathcal{V}_1}\Big(1+\kappa^{\frac{1}{2}}\epsilon_{\mathcal{V}_2}\Big), \quad (45)$$

for all $t \in \overline{\mathbb{R}}_+$. The performance bound in (45) implies that the distance between $x(t)$ and $x_{\mathrm{r_i}}(t)$ can be made arbitrarily small in transient time by resorting to a high learning rate γ, similar to Remark 2.1 for the standard model reference adaptive control scheme. However, as we see in the next section, by increasing κ, we make the distance between $e(t)$ and $e_{\mathrm{L}}(t)$ sufficiently small in transient time, and hence, a high learning rate γ subject to a high κ does not yield to signals with high-frequency oscillations. Finally, it should be also noted from (45) that keeping γ constant but increasing κ may result in a larger distance between $x(t)$ and $x_{\mathrm{r_i}}(t)$, and therefore, both should be increased simultaneously in order to keep this distance consistent during transient time.

D. Suppressing High-Frequency System Error Dynamics

We now show that the high-frequency content of the system error

$$e_{\mathrm{H}}(t) \triangleq e(t)-e_{\mathrm{L}}(t), \quad (46)$$

can be effectively suppressed as one increases κ design parameter of the modified reference system (26). To see this, let $\varepsilon \triangleq \kappa^{-1}$. Then, (27) and (24) can be equivalently written as

$$\varepsilon\dot{e}(t) = \varepsilon A_{\mathrm{r}}e(t)-\varepsilon B\Lambda\tilde{W}^{\mathrm{T}}(t)\sigma\big(x(t)\big)-\big(e(t)-e_{\mathrm{L}}(t)\big), \quad (47)$$

$$\dot{e}_{\mathrm{L}}(t) = A_{\mathrm{r}}e_{\mathrm{L}}(t)+\eta\big(e(t)-e_{\mathrm{L}}(t)\big). \quad (48)$$

Since setting $\varepsilon = 0$ results in $0 = e(t) - e_{\mathrm{L}}(t)$ and $\dot{e}_{\mathrm{L}}(t) = A_{\mathrm{r}}e_{\mathrm{L}}(t)$, then the system given by (47) and (48) is said to be the singularly perturbed model form, where $e(t) = e_{\mathrm{L}}(t)$ captures the isolated root. To shift the quasi steady-state of $e(t)$ to the origin, consider $e_{\mathrm{H}}(t) = e(t)-e_{\mathrm{L}}(t)$ as a change of variables, which yields

$$\frac{\mathrm{d}e_{\mathrm{H}}(\tau)}{\mathrm{d}\tau} = -e_{\mathrm{H}}(\tau), \quad e_{\mathrm{H}}(0) = e_{\mathrm{H_0}}, \quad (49)$$

where τ is related to the original t through

$$\tau = t/\varepsilon. \quad (50)$$

As a consequence of Theorem 11.2 in [7], it can be further shown that

$$e_{\mathrm{H}}(t,\kappa) = \mathrm{e}^{-\kappa t}e_{\mathrm{H_0}} + \mathcal{O}(\kappa^{-1}), \quad (51)$$

holds for a sufficiently high κ, where

$$e_{\mathrm{H_0}} \triangleq e(0)-e_{\mathrm{L}}(0) = e_0. \quad (52)$$

That is, the transient high-frequency content of the system error $e_{\mathrm{H}}(t)$ vanishes in a fast manner for a sufficiently high κ.

IV. Illustrative Example

Consider the nonlinear dynamical system representing a controlled wing rock aircraft dynamics model given by

$$\begin{bmatrix}\dot{x}_{\mathrm{p}_1}(t)\\ \dot{x}_{\mathrm{p}_2}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_{\mathrm{p}_1}(t)\\ x_{\mathrm{p}_2}(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}\big[\Lambda u(t)+\delta_{\mathrm{p}}(t,x_{\mathrm{p}}(t))\big], \quad (53)$$

where $x_{\mathrm{p}_1}(0) = 0$, $x_{\mathrm{p}_2}(0) = 0$, x_{p_1} represents the roll angle in radians, and x_{p_2} represents the roll rate in radians per second. In (53), $\delta_{\mathrm{p}}(t, x_{\mathrm{p}})$ and Λ represent uncertainties of the form

$$\delta_{\mathrm{p}}(t,x_{\mathrm{p}}) = \alpha_1\sin(t)+\alpha_2x_{\mathrm{p}_1}+\alpha_3x_{\mathrm{p}_2}+\alpha_4|x_{\mathrm{p}_1}|x_{\mathrm{p}_2} +\alpha_5|x_{\mathrm{p}_2}|x_{\mathrm{p}_2}+\alpha_6x_{\mathrm{p}_1}^3, \quad (54)$$

and $\Lambda = 0.75$, where α_i, $i = 1,\ldots,6$, are unknown parameters. For our numerical example, we set $\alpha_1 = 0.25$, $\alpha_2 = 0.5$, $\alpha_3 = 1.0$, $\alpha_4 = -5.0$, $\alpha_5 = 5.0$, and $\alpha_6 = 10.0$. We chose $K = [2.0,\ 2.0,\ 1.0]$ for the nominal controller design. For the proposed adaptive control architecture (Theorem 7.1),

$$\sigma(x) = \big[1,\ x_{\mathrm{p}_1},\ x_{\mathrm{p}_2},\ |x_{\mathrm{p}_1}|x_{\mathrm{p}_2},\ |x_{\mathrm{p}_2}|x_{\mathrm{p}_2},\ x_{\mathrm{p}_1}^3,\ x^{\mathrm{T}}\big]^{\mathrm{T}}, \quad (55)$$

is chosen as the basis function and we set $R = I_3$. Figs. 3–4 present the results, where measurement noise is added to the state vector of (53) and α_1 is set from 0 to 0.25 both at $t = 45$ seconds (that is, exogenous time-varying disturbance $\sin(t)$ is added at $t = 45$ seconds) for all cases. Here, our aim is to follow a square-wave roll angle command $c(t)$.

Fig. 3 shows the closed-loop system performance of the standard model reference adaptive control approach ($\gamma = 500$, $\kappa = 0$, and $\eta = 0$). Even though we achieve a satisfactory command following performance with this approach, as discussed in Remark 3.1, its control performance is unacceptable due to high-frequency oscillations and measurement noise amplification.

Next, we show the closed-loop system performance of the proposed model reference adaptive control approach ($\gamma = 500$, $\kappa = 100$, and $\eta = 5$) in Fig. 4. In particular, we achieve a satisfactory command following performance similar to the case in Fig. 3. However, the control response of our approach is clearly superior as compared to the control response of the standard model reference adaptive control approach in Fig. 3. This is expected from the proposed theory, and hence, the control response of the proposed approach neither has high-frequency oscillations nor high measurement noise amplification.

V. CONCLUSION

We presented a new reference system for model reference adaptive controllers in order to improve the transient performance. Specifically, by utilizing singular perturbation theory, it is shown that the proposed reference system allows to limit the frequency content of the system error dynamics, and hence, yields fast adaptation without incurring high-frequency oscillations in the transient performance.

VI. ACKNOWLEDGMENT

The authors wish to thank Eugene Lavretsky from the Boeing Company for his constructive suggestions.

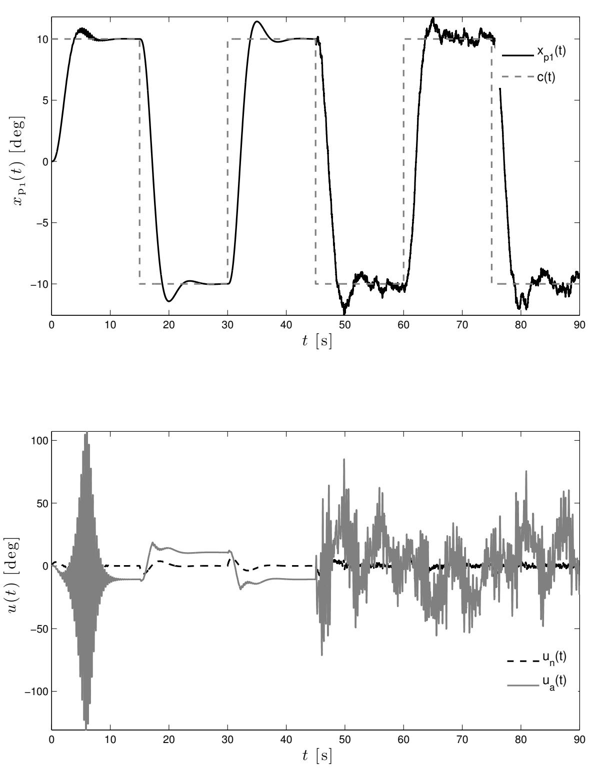

Fig. 3. Command following performance for the standard model reference adaptive control approach ($\gamma = 500$, $\kappa = 0$, and $\eta = 0$).

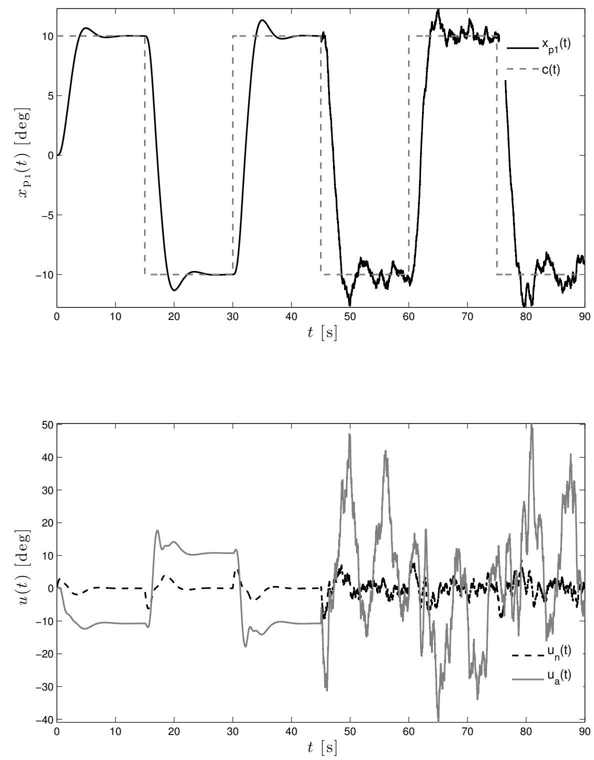

Fig. 4. Command following performance for the proposed model reference adaptive control approach ($\gamma = 500$, $\kappa = 100$, and $\eta = 5$).

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