Comparison of Robust and Probabilistic LMI-Based Design of Adaptive Flight Controllers with Uncertain Input Dynamics

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Adaptive control-based schemes have been implemented to date in a variety of different applications. However, the ability to obtain a predictable transient closed-loop performance in adaptive systems is still a challenging problem from a verification and validation point of view. To face this problem we have recently introduced an analysis and design framework for adaptive control systems in the presence of bounded uncertainty and bounded adaptive control (the boundedness can be enforced, for instance, by a parameter projection mechanism) showing that the transitory performance of a MRAC system can be expressed, analyzed, and optimized via a convex optimization formulation based on Linear Matrix Inequality (LMI) requirements. A key feature of this framework is that it is possible to tune the adaptive control parameters rigorously so that the tracking error of the closed-loop system evolves within an a priori specified region of the error space whose size can be minimized by selecting a suitable cost function.

One drawback of this approach is the possible conservatism of the results. In fact, as with any robust control problem, the design philosophy is to guarantee the performance for all the set of allowed uncertainties. The consequence of this fact is that the robust LMIs constraints derived in this context may lead to conservative conditions due to unavoidable matrix majorations requested in the derivation of the robust LMI conditions.

To overcome these limitations in this study we propose a novel stochastic analysis and design framework for MRAC systems where the uncertain parameters along with the adaptive control signal are considered as random variables. This brings the important advantage that the (stochastic) LMI conditions that define the performance requirements can be immediately derived from Lyapunov analysis without the need of matrix majorations and of the introduction of auxiliary variables. In turn, this leads to the fact that the stochastic LMI conditions are less conservative. In this paper we compared the robust worst-case framework with the novel stochastic framework. The (previous) robust and the (novel) probabilistic convex optimization approaches were both applied for the optimized design of minimum size tracking error invariant sets for a MRAC control systems in the presence of matched and of input uncertainty acting on the actuator dynamics. The comparative study was performed using the short period longitudinal dynamics of an F-16 aircraft model.
I. Introduction

The problem of characterizing the transitory response and robustness properties of adaptive control systems in the presence of uncertainties has been a very active research area for years\textsuperscript{2,3,5}. The difficulty in predicting the transient response originates from the inherent nonlinearity in Model Reference Adaptive Control (MRAC) schemes. This topic is particularly relevant in the context of safety critical applications such as flight control systems.

For instance, it is extremely difficult to characterize the transient response of adaptive flight control systems in the presence of time delays, unmodelled dynamics, disturbances, and unmatched uncertainties. To address the above mentioned issues some robust modifications of the basic adaptation mechanisms have been proposed\textsuperscript{6} to prevent the parameter drift caused by modelling errors and to guarantee uniform ultimate boundedness (UUB) of the tracking error in a residual set whose size can be reduced by increasing the adaptation gain.

Despite the recent progresses reported in the literature\textsuperscript{5,6,7}, the research oriented toward the development of general analysis and design frameworks for uncertain adaptive control systems is still open\textsuperscript{8}.

Recently, the authors proposed a general framework\textsuperscript{9,10,11} to addresses the problem of quantifying performance and operative regions of uncertain adaptive control systems using Linear Matrix Inequalities (LMIs) constraints. The key aspect of the methodology is that the dynamics of an uncertain adaptive control system (for instance featuring a parameter projection adaptation law) is traced back to the dynamics of a linear system perturbed by bounded uncertainties. The advantage of this framework is that quantitative performance and robustness requirements can be expressed, analyzed and optimized performing a convex optimization with LMIs constraints. This approach provides a constructive method for the design of UUB regions for the tracking error with a desired shape. The methodology is based on the design of ellipsoidal invariant sets that naturally originates from the employment of quadratic control Lyapunov functions\textsuperscript{12,13}.

One drawback of this approach is the possible conservatism of the results. In fact, as with any robust control problem, the design philosophy is to guarantee the performance for all the set of allowed uncertainties. The consequence of this fact is that the LMIs constraints derived in this context may lead to conservative conditions. In the works\textsuperscript{9,10,11} the conservatism originates from the unavoidable matrix majorations requested in the derivation of LMI conditions that should hold for all the possible uncertainties. This approach entails also the introduction of a number of auxiliary optimization variables in the LMIs that makes the optimization problem more involved.

To overcome these limitations in the present study we focused the attention toward a stochastic optimization framework\textsuperscript{14,15} where the uncertain parameters along with the adaptive control signal are considered as random variables with the important advantage that the (stochastic) LMI conditions that define the performance requirements can be immediately derived from Lyapunov analysis without the need of matrix majorations and of the introduction of auxiliary variables.

The idea of considering the probabilistic framework was inspired by papers like\textsuperscript{16} where a probabilistic convex optimization design was applied to a Linear Parameter Varying (LPV) system where an uncertain linear system is ‘perturbed’ by a bounded scheduling signal that is assumed measurable but a-priori unknown. In\textsuperscript{16} the uncertainty and the scheduling signals are considered as random variables having specific probability density functions and a stochastic convex optimization problem was set-up to compute a probabilistic solution\textsuperscript{17}.

We found that this approach can be adapted also in the context of adaptive MRAC systems where the bounded adaptive control contribution (whose boundedness can be enforced, for instance, by a parameter projection mechanism) takes the place of the bounded scheduling parameter.

The main contribution of this paper is, thus, the reformulation of the LMI requirements for the design of tracking error invariant sets in the context of a probabilistic convex optimization framework for an MRAC control system. The so-called Scenario Approach\textsuperscript{18} was then employed to perform the convex optimization in the new stochastic context.

The (previous) robust and the (novel) probabilistic approaches were both applied for the optimized design of minimum size tracking error invariant sets for a MRAC control systems in the presence of matched and of input uncertainty acting on the actuator dynamics. The comparative study was carried out considering the short period longitudinal dynamics of an F-16 aircraft model.
II. MRAC of an aircraft with uncertain input dynamics

In this section we define the MRAC architecture and the modelling assumptions considered in the study. Consider a single input uncertain dynamical system:

\[ \dot{x} = A_x x + B_x \left( \Delta(x) - u_{\mu} \right) \]  

where \( x(t) \in \mathbb{R}^n \) is the state (accessible) vector, \( A_x \in \mathbb{R}^{n \times n} \) and \( B_x \in \mathbb{R}^{n \times 1} \) are nominal system matrices, \( \Delta(x) \in \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a matched system uncertainty and the control signal \( u_{\mu}(t) \) is the output of an uncertain linear actuator \( u_{\mu}(s) = G(s)u(s) \) whose states are not accessible for feedback. In this study the uncertainty is assumed linearly parameterized as follows,

\[ \Delta(x) = W^T x \]  

where \( W \in \mathbb{R}^{m \times 1} \) is an unknown constant vector. To address command following let \( r(t) \in \mathbb{R} \) be a given bounded and smooth command signal such that

\[ |r(t)|^2 \leq r_0^2 \]  

where \( r_0 \) is a design parameter. The feedback control law is composed of three contributions

\[ u(t) = u_n(t) + u_e(t) + u_{\text{ad}}(t) \]  

The nominal linear control law \( u_n(t) \) is given by

\[ u_n(t) = -K_0 x(t) - K_r r(t) \]  

where the vector \( K_0 \in \mathbb{R}^{n \times n} \) is chosen such that \( A = A_x + B_x K_0 \) is Hurwitz and \( K_r \in \mathbb{R} \) is selected to guarantee a desired gain between the reference and the performance output so that transitory and steady state performance is satisfied. Typically, the design is carried out using the LQR or the pole placement technique. The second control contribution \( u_e(t) \) depends on the tracking error \( e(t) = x(t) - x_{\text{ref}}(t) \) (where \( x_{\text{ref}}(t) \) is the state of the reference model) and it is defined as

\[ u_e(t) = K_r e(t). \]  

The gain matrix \( K_r \) will be computed in the optimized design phase and will be used, as additional feedback contribution to enforce tracking error requirements in the presence of uncertainty. The last term \( u_{\text{ad}}(t) \) is the adaptive control feedback contribution and it is defined as

\[ u_{\text{ad}}(t) = \hat{W}^T(t)x(t) \]  

where \( \hat{W}(t) \) is an adaptive estimate of the unknown vector \( W \). The overall control signal \( u(t) \) in (4) is filtered by the actuator dynamics that is modelled as an uncertain gain first order low-pass filter with time constant \( \tau = x^{-1} \), that is

\[ \dot{u}_f = -\lambda_u f + \lambda_x \left[ \hat{W}^T x - K_0 x - K_r r + K_r e \right] \]  

\[ u_{f_0} = (1 + \Lambda_u) u_f \]  

where \( \Delta_u \) is an unknown bounded gain that models possible uncertainty on the actuator effectiveness. The gain \( \Delta_u \) is parameterized as

\[ \Delta_u = \Delta_u^\text{max} \delta_u \]  

where \( |\Delta_u^\text{max}| \leq 1 \) and \( \delta_u \) is the maximum uncertainty allowed for the control efficiency. The reference model is defined as

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\[ x_m = (A_x + B_n K_0)x_m + B_r K_f r = A x_m + B r \]  

(10)

Defining the adaptive weight estimation error as \( \hat{W}(t) = \dot{W}(t) - W \), and considering the system and reference model dynamics (1) and (10) respectively, the resulting tracking error dynamics is

\[ \dot{e} = A e - B_o (\bar{W}^T x + u_{f_0} - \bar{W}^T x + K_o x + K_f r) \]  

(11)

A. The adaptation algorithm

To guarantee the boundedness of the estimated vector \( \hat{W} \) while avoiding the drift of the parameters due to unmodelled actuator dynamics, we adopted a parameter projection based adaptation law, featuring a sigma modification term as proposed in \(^{19}\)

\[ \hat{W} = \Gamma \text{proj}\left( \hat{W}, x e^T P B_o - \sigma \hat{W} \right) \]  

(12)

where \( \Gamma \) is the parameter adaptation rate (a scalar in this study), \( \sigma \) is the sigma modification coefficient and the matrix \( P > 0 \) will be computed in the following optimized design. The sigma modification term in (12) is used to damp out possible chattering in the control signal that could be induced in case of high gain adaptation. By construction, the projection operator ensures the boundedness of \( \hat{W}(t) \) within a hyper-sphere domain such that \( \| \hat{W}(t) \| \leq W_0^2 \) where \( W_0 \) is a user defined upper bound of the norm of the matched uncertainty. In the following robust worst-case analysis it will be useful to consider \( \hat{W}(t) \) as a norm bounded uncertainty expressed as

\[ \hat{W}(t) = \Delta(t) W_0 \quad \Delta(t)^T \Delta(t) \leq 1 \]  

(13)

Assuming that the unknown vector \( W \) is internal to the parameter adaptation domain (that is \( \| W \| \leq W_0^2 \)), it is immediate to show that the parameter estimation error is bounded as follows

\[ \hat{W}^T(t) \hat{W}(t) \leq 4 W_0^2 = \delta_0^2 \]  

(14)

The bounds (14), (9) and (3) will be used in the next sections to characterize the region of variation of the uncertainty and of the adaptive control signal.

III. Performance characterization via LMIs

In this section a general method for characterizing invariant regions and performance requirements for the tracking error of the MRAC system introduced in section II is presented. The method is based on the design of robust ellipsoidal invariant sets generated by quadratic Lyapunov functions \(^{11}\). The analysis of the closed loop adaptive system is carried out using the standard quadratic Lyapunov function defined as follows

\[ V(e, u_f, \hat{W}, x_m) = e^T P e + u_f^T P_1 u_f + \hat{W}^T \Gamma^{-1} \hat{W} + x_m^T P_2 x_m \]  

(15)

where \( P \) and \( P_2 \) are symmetric positive definite matrices and \( P_1 \) and \( \Gamma \) are positive scalars. The ellipsoidal set \( \Omega \) generated by the level curve \( V(e, u_f, \hat{W}, x_m) = 1 \) is defined as

\[ \Omega = \{ e, u_f, \hat{W}, x_m \ | e^T P e + u_f^T P_1 u_f + \hat{W}^T \Gamma^{-1} \hat{W} + x_m^T P_2 x_m \leq 1 \} \]  

(16)

A. Requirements on the Lyapunov function

We require the Lyapunov function (15) to be positive definite, that is \( V(e, u_f, \hat{W}, x_m) > 0 \) for all \( (e, u_f, \hat{W}, x_m) \neq (0, 0, 0, 0) \). This requires the satisfaction of the following LMIs:

\[ P > 0, \ P_1 > 0, \ P_2 > 0, \ \Gamma^{-1} > 0 \]  

(17)

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B. Requirements on the shape of the tracking error invariant set

For tracking error performance analysis it is important to constrain the shape of the set $\Omega$ along the tracking error components. For this purpose we define the subset $\Omega_e = \{e| e^TPe \leq 1\}$ as the projection of the set $\Omega$ in the tracking error variables subspace. We require the ellipsoid $\Omega_e$ to be contained in a polyhedron $\Pi_{e\rho}$ that is defined as follows

$$\Pi_{e\rho} = \{e| e_i(t) \leq \rho_i e_{Mi}\} \quad i = 1,...,n$$

(18)

where the $e_{Mi}$ are user defined reference performance bounds for the components of the error and the $\rho_i$ are (scaling) parameters that will be determined in the optimized design. For the problem under investigation, the main objective of the design will be the reduction of the size of the set $\Pi_{e\rho}$ in the presence of bounded uncertainties. It can be shown\(^{13}\) that the condition $\Omega_e \subseteq \Pi_{e\rho}$ holds if the following LMI's are satisfied

$$\begin{bmatrix} P & g_i \\ (g_i)^T & \rho_i^2 e_{io}^2 \end{bmatrix} > 0 \quad i = 1,...,n$$

(19)

where $g_i = [0,...,1,...,0]^T$ is the $i^{th}$ versor of $R^n$.

C. Requirements on the initial condition

Considering the initial conditions we require the set $\Omega_e$ to contain an a-priori specified initial condition subset $\Omega_0$ for the tracking error components, that is we require that $\Omega_0 \subseteq \Omega_e$. In this study we considered an ellipsoidal initial condition subset that is defined as $\Omega_0 = \{e| e^T P_o e \leq 1\}$ where $P_o$ is a symmetric positive definite shape matrix to be defined at design level. Applying the S-procedure\(^{13}\) it can be shown that the containment of the ellipsoid $\Omega_0$ in the ellipsoid $\Omega_e$ holds if the following LMI requirement is satisfied:

$$\begin{bmatrix} -P + \tau P_o & 0 \\ 0 & 1 - \tau \end{bmatrix} < 0$$

(20)

where $\tau$ is an arbitrary positive scalar. In this study $P_o$ is defined as

$$P_o = \begin{bmatrix} e_{io}^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & e_{io}^2 \end{bmatrix}^{-1}$$

(21)

where the $e_{io}$ are design values defining the length of the semi-axis of the ellipsoid $\Omega_0$ along the $i$-th error component.

D. Requirement on norm $\|K_r\|$ 

In the optimized design (as clarified shortly), it is important to have the control of the norm of the gain matrix $K_r$ in (6). This was achieved by constraining the norm of the matrix $\bar{K}_r = K_r^T P_1$ (in this study $P_1$ is scalar) such that $\bar{K}_r^T \bar{K}_r < \eta$, where $\eta>0$ is an optimization parameter. This constraint is equivalent to the following LMI\(^{13}\) requirement

$$\begin{bmatrix} \eta & \bar{K}_r \\ \bar{K}_r^T & I \end{bmatrix} > 0$$

(22)

The above performance requirements expressed by the MI constraints (17), (19), (20) and (22) will be used both in the Robust Optimization (RO) design and in the Probabilistic Optimization (PO) design that will be introduced in the next sections.
IV. Performance invariant set design: Robust (worst-case) LMI formulation

In this section a LMI formulation that ensures the invariance of the set Ω defined in (16) that is compatible with a RO worst case-design is presented. The RO design is based on the concept of robust positive invariance for the set Ω. For the problem under investigation the definition is particularized as follows

Definition 1: The set Ω is said to be robustly positively invariant for the error dynamics (11), adaptation law (12), reference model (10) and actuator dynamics (8) if, for any initial state ∈ Ω, the trajectory of the adaptive system remains in Ω for all t, for any bounded input satisfying (3), for any actuator uncertainty Δu satisfying (9) and for any bounded adaptation weight error satisfying (14).

A sufficient condition for the robust positive invariance of the set Ω is given by the following theorem.

Theorem 1: Consider the closed loop error dynamics (11), the adaptation law (12), the reference model (10), the actuator dynamics (8) and the bounded reference satisfying (3), actuator uncertainty Δu satisfying (9) and bounded adaptation weight error satisfying (14). If there exist positive scalars α, β, γ, ε1, ε2, εb, symmetric positive define matrices P, P1, P2, T−1 and a matrix Kc satisfying the constraint in (23) then, Ω is robustly positively invariant.

\[ \begin{bmatrix} A^T + P + PA + εI + λe_1 I + \frac{−PB}{−PB} \quad \frac{−PB}{−PB} \quad \frac{−PB}{−PB} \quad \frac{−PB}{−PB} \quad \frac{−PB}{−PB} \quad 0 \quad 0 \quad PBW_0 \quad 0 \quad 0 \quad PB_εb \\ +βP_1 + PB_0 - K_ε^T B_ε^T P_1; \quad 0 \quad −PB \quad 0 \quad PBW_0 \quad 0 \quad 0 \quad PB_εb \end{bmatrix} \]

\[<0\]  

(23)

The complete proof of this theorem was given in.

Remark-1: An important aspects of the above result is that it was derived assuming "worst-case" uncertainty that is condition (23) holds for any admissible bounded reference signal such that \( r^T r ≤ ε_r^2 \), for any admissible bounded adaptive parameter estimation error \( W^T \hat{W} < 4W_0^2 = δ_u^2 \) and for any bounded actuator gain uncertainty \( δ_u^2 \leq δ_u^2 \). A possible drawback of condition (23) it that it was derived following a series of successive uncertain matrix majorizations that unavoidably introduce conservatism in the solution. In many practical situations worst-case uncertainty is extremely unlikely to occur since it is not representative of the actual uncertainty acting on the system.

A. Multiobjective optimized invariant set design

Since the design requirements in (17), (19), (20) and (22) along with the robust invariance condition (23) are all expressed in terms of LMIs, we can formulate the analysis and design problem as a convex optimization with a linear cost function (to be defined) whose optimization variables are the free parameters involved in the above LMIs. In this study a multiobjective cost function was chosen with the purpose of minimizing the size of the scaled error polyhedron Πε. While producing a minimal authority controller Kc. This objective is quantified by the linear cost function:

\[ J = \sum_{i=1}^{n} ρ_i \cdot \xi_i + χ \cdot η - ζ \cdot P_i \]  

(24)

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where the coefficients $\xi_i$ weight the importance of the tracking error components, $\chi$ weights the control authority of $K_c$ and the last coefficient $\zeta$ is used to have the control on $P_i$ in order to avoid very small values for this parameter since $K_c = \hat{K}_c^T / P_i$

To formalize the optimization problem we are requested to define the set of the optimization variables named $\theta_{ro}$. In this study $\theta_{ro}$ includes all the free parameters involved in the LMIs (17), (19), (20), (22) and (23), that is $\theta_{ro} = [P_i, P_2, \Gamma, \alpha, \beta, \gamma, \epsilon, \epsilon_1, \epsilon_2, \epsilon_3, \rho_1, \rho_2, ..., \rho_m, \hat{K}_c, \eta]$. For conciseness we also collect the sets of LMIs (17), (19), (20), (22) and (23) is a vector of convex LMI conditions named $f_{ro}(\theta_{ro})$. With the condition $f_{ro}(\theta_{ro})<0$ we shall conventionally mean that all the inequality constraints in $f_{ro}(\theta_{ro})$ are satisfied. The RO problem is then formalized as

$$(\text{RO}): \text{minimize } J(\theta_{ro}) \text{ s.t. } f_{ro}(\theta_{ro})<0 \quad (25)$$

Note that the (RO) problem (25) is not linear due to the presence of the product between some of the optimization variables $\theta_{ro}$ in the LMIs in $f_{ro}(\theta_{ro})$. A practical solution to this problem will be given in section IV-B.

Remark-2: It should be observed that the LMIs conditions in $f_{ro}(\theta_{ro})$ do not depend on uncertain parameters but depends on deterministic upper bounds for these parameters and on some auxiliary variables.

B. Solving procedure for the (RO) problem

The engineering approach used to solve the nonlinear optimization problem (25) is to fix a subset of the optimization variables in $\theta_{ro}$ so that the reduced dimension problem turns out to be linear in the remaining variables. Then, the resulting linear optimization is repetitively solved by performing a discretized grid search on the subspace of the fixed (grid) variables. Finally, the solution leading to the minimum value of $J$ is considered as the suboptimal solution. The selection of the subset of the grid variables depends on the particular design aspect under investigation.

Remark-3: The proposed procedure for the numerical solution of the nonlinear problem (25) requires a repeated solution of a linear problem on a grid. This brings along the well-known problems of nonlinear optimization such as the locality of the suboptimal solution, grid size and resolution. The availability of efficient linear programming solvers for large scale optimization enables today the very fast exploration of large domains in reasonable time.

V. Probabilistic formulation of the robust convex optimization problem

In contrast to the RO worst-case design approach in the probabilistic framework the design is considered successful even if it does not guarantee, a-priori, the performance for the whole class of the allowed uncertainties. As in the RO optimization framework the design requirements are translated in a set of MI constraints of the form $f(\theta_{ro}, q)<0$ where $\theta_{ro}$ represents the set of the free design parameters in the stochastic formulation and $q$ represents a set of random parameters. More precisely in our study the vector $q$ will collect the bounded uncertainties, the bounded refer input command and the bounded adaptive control signals whose probability density functions are defined over a bounded subset named $Q$. Since the LMI constraints are allowed to be stochastic, in this framework we have to solve a stochastic convex optimization problem.

The optimization approaches followed to manage probabilistic convex optimization problems are typically based on uncertainty randomization and are usually called randomized algorithms. That is, assuming that $q$ is a random vector with given probability density functions, an optimized solution is derived utilizing a finite number of random samples $q(i)$ of $q$. These algorithms provide an optimized solution that probabilistically satisfies the set of MI constraint $f(\theta_{ro}, q)<0$. In other words, a certain probability of violation $V(\theta_{ro}, q)$ is associated to an optimized solution $\theta_{ro}$, but this probability may be suitably bounded by a given (probabilistic) accuracy $\epsilon \in (0,1)$ and by a given confidence $\delta \in (0,1)$.

A. The Scenario approach for stochastic optimization

In this work we employed the randomized optimization algorithm known as the Scenario approach that is defined. This approach deals with a general robust convex optimization problems of the form

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\[
\text{minimize } J(\theta) \text{ s.t. } f(\theta, q) < 0 \text{ } \forall q \in Q
\]

(26)

Following the probabilistic approach a probabilistic description of the uncertainty is considered, that is we formally assume that \( q \) is a random variable with a given probability density function defined over \( Q \). Then, \( N \) independent identically distributed (iid) samples \( q(1), \ldots, q(N) \) are extracted according to the probability distribution of the uncertainty to form the multi-sample \( q=[q(1), \ldots, q(N)] \). These samples constitutes the so called scenarios. The scenario optimization (SO) problem is thus based on the \( N \) samples of the uncertainty. In other words we consider the following problem

\[
(\text{SO}): \text{minimize } J(\theta_{so}) \text{ s.t. } f_{so}(\theta_{so}, q(i)) \text{ } i=1,\ldots,N
\]

(27)

In (27) we have adopted the subscript 'so' to emphasize the difference between the SO optimization problem in (27) and the RO in (25). The SO problem may be interpreted as a probabilistic relaxation of the robust optimization problem in (26) since it deals only with a subset of the (infinite) constraints considered in (26), according to the probability distribution of the uncertainty\(^\text{17}\).

However, under mild assumptions, by suitably choosing \( N \), this approximation may in practice become negligible. Specifically, \( N \) can be selected depending on the level of "risk" of constraint violation that the user is willing to accept. The constraints violation probability for each value of \( \theta_{so} \) is defined as

\[
\text{Viol}(\theta_{so}) = \Pr(q \in Q : f(\theta_{so}, q) \geq 0)
\]

(28)

A fundamental result of the scenario approach method is the explicit definition of the sample size \( N \) that should be used in the optimization (26) in order to guarantee a solution with a desired accuracy \( \epsilon \) and confidence \( \delta \). This is formally expressed by the following theorem.

**Theorem-2:** Let the convexity assumption hold and let \( n_0 \) be the number of optimization variables in the \( \theta_{so} \) vector. Suppose that \( N > n_0, \) and, \( \epsilon \) and \( \delta \in (0, 1) \) satisfy the inequality\(^\text{18}\)

\[
\left( \frac{N}{n_0} \right) (1-\epsilon)^{N-n_0} \leq \delta
\]

(29)

then the probability that \( \text{V}(\theta_{so}) > \epsilon \) is at most \( \delta \).

The bound in (29) is important because it provides an implicit relation between \( N, n_0, \epsilon \) and \( \delta \). This relation can be made explicit to derive the sample complexity of the scenario approach. In\(^\text{23}\) it was shown that the sample complexity \( N \) is proportional to \( 1/\epsilon \) for a fixed \( \delta \). In particular, it was shown that that, for given \( \epsilon \) and \( \delta \in (0, 1) \), (29) holds if

\[
N \geq \frac{2}{\epsilon} \left[ \ln \frac{1}{2\delta} + 2n_0 + \frac{2n_0^2}{\epsilon} \ln 4 \right]
\]

(30)

This relation will be exploited in the experimental section of the paper to compute the sample size \( N \) of the uncertainty to be used in the scenario optimization to guarantee a given \( \epsilon \) and \( \delta \).

**VI. Invariant set design: Probabilistic LMI formulation**

The purpose of this section is to formulate (a novel) invariance LMI condition for the set \( \Omega \) defined in (16) in a probabilistic context that is suitable for the probabilistic optimization. In more details the worst-case LMI condition (23) will be substituted with a probabilistic (less conservative) LMI condition, while the LMIs (17), (19), (20), (22) that do not depend on the uncertainty remain unchanged.

To derive a probabilistic LMI condition for the invariance of the set \( \Omega \), we restart from the computation of the time derivative of the Lyapunov function defined in (15). Substituting the dynamics (8), (11) and (12) in \( \dot{V}(t) \) (and ex-

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no property of projection operator: \( \hat{W} \left[ \text{Proj} \left[ \hat{W}, x e^T PB - \sigma \hat{W} \right] - x e^T PB - \sigma \hat{W} \right] \leq 0 \), it is not difficult to show that:

\[
\dot{V} \leq e^T (A^T P + PA)e^{-2\sigma \hat{W}^T \hat{W} - 2e^T PB \partial_x (u_j + \Delta u_j) + 2e^T PB \hat{W}^T (e + x_m) - 2e^T PB_o K_o (e + x_m) - 2e^T PB r \nonumber
+ \gamma_m (A^T P_1 + P_2 A) x_m + 2\gamma_m P_2 Br - 2\gamma_m P_1 u_j - 2\gamma_m P_1 K_r + 2\gamma_m P_1 \hat{W}^T (e + x_m) - 2\gamma_m P_1 K_o (e + x_m) + 2\gamma_m P_1 K_r e
\]

(31)

Now, in order to guarantee the invariance of \( \Omega \), the condition \( \dot{V}(t) \leq 0 \) have to be verified on the boundary \( \partial \Omega \) and in the region external to \( \Omega \) that is defined as

\[
e^T P e + u_j^T P u_j + \hat{W}^T \Gamma^{-1} \hat{W} + \gamma_m P_2 x_m \geq 1
\]

(32)

The condition \( \dot{V}(t) \leq 0 \) with constraint (32) can be transformed in an equivalent single LMI using the S-procedure. Applying the S-procedure, it results that the above conditions are satisfied if there exist a positive coefficient \( \beta \) such that:

\[
e^T (A^T P + PA) e^{-2\sigma \hat{W}^T \hat{W} - 2e^T PB \partial_x (u_j + \Delta u_j) + 2e^T PB \hat{W}^T (e + x_m) - 2e^T PB_o K_o (e + x_m) - 2e^T PB r \nonumber
+ \gamma_m (A^T P_1 + P_2 A) x_m + 2\gamma_m P_2 Br - 2\gamma_m P_1 u_j - 2\gamma_m P_1 K_r + 2\gamma_m P_1 \hat{W}^T (e + x_m) - 2\gamma_m P_1 K_o (e + x_m) + 2\gamma_m P_1 K_r e
\]

\[
\beta \left( e^T P e + u_j^T P u_j + \hat{W}^T \Gamma^{-1} \hat{W} + \gamma_m P_2 x_m \right) \leq 0
\]

(33)

Rearranging (33) as function of the vector \( z = [e^T, u_j^T, x_m^T, 1]^T \) it can be shown that (33) can be expressed in quadratic for as \( z^T z \leq 0 \) where

\[
Z = \begin{bmatrix}
A^T P + PA - 2PB_o K_o & -PB_o (1 + \Lambda_u) & -PB_o P_1 & -PB_o K_0 & -PB_o r \\
\beta P + PB_o \hat{W}^T & +\lambda (P_1 \hat{W} - K_o P_1 + K_o P_1) & +PB_o \hat{W}^T & -PB_o K_0 & -PB_o r \\
* & -2P_1 \lambda + \beta P_1 & +\lambda P_1 \hat{W} - P_1 K_0 & -\lambda PB_o r \\
* & * & A^T P_2 + P_2 A + \beta P_2 & -P_2 B_r r \\
* & * & * & -2\hat{W}^T \hat{W} \sigma + \\
& & & \beta \hat{W}^T \Gamma^{-1} \hat{W} - \beta
\end{bmatrix}
\]

(34)

A sufficient condition for (34) to hold is that the matrix \( Z \) is negative definite, that is

\[
Z(\Delta_u, \hat{W}, \hat{W}, r) < 0
\]

(35)

The above inequality is interpreted as a stochastic LMI that is the variables \( q_0 = [\Delta_u, \hat{W}, \hat{W}, r] \) are here considered (only at the design stage) as random variables that are defined over suitable and know bounded sets. In more details the stochastic variables are characterized as follows

\[
r \in Q_r = \{[-\delta_r, +\delta_r] \} \text{ uniform distribution}
\]

\[
\Delta_u \in Q_{\Delta_u} = \{[-\delta_u, +\delta_u] \} \text{ uniform distribution}
\]

\[
\hat{W} \in Q_{\hat{W}} = \{\hat{W}^T \hat{W} < W_0^2 \} \text{ uniform distribution}
\]

\[
\hat{W} \in Q_{\hat{W}} = \{\hat{W}^T \hat{W} < W_0^2 \} \text{ uniform distribution}
\]

(36)

Considering the above definitions it results that all the entries of LMI (34) are multi-affine in the stochastic variables defined in (36) except the term \( \beta \hat{W}^T \hat{W} \sigma \hat{W} \) that is indeed quadratic in \( \hat{W} \). To overcome this quadratic dependence we define a new auxiliary stochastic scalar variable \( \bar{W} = \hat{W}^T \hat{W} \) as follows

\[
\bar{W} \in Q_{\bar{W}} = \{0, 2W_0^2 \} \text{ uniform distribution}
\]

(37)
This new auxiliary variable introduces some conservatism in the design but allows the inequality (35) to be multi-affine in \(q=[\Delta_0, W, W, r, W]\). This fact will be exploited in the experimental part of the paper for comparing stochastic and robust worst-case solutions. The domain of definition of the random vector \(q\) is defined as \(Q=Q_{aw} \cup Q_W \cup Q_k \cup Q_r\).

The Scenario approach requires the computation of \(N\) LMI constraints of the form

\[
Z(\Delta_{so}(i), W(i), r(i), W(i)) < 0 \quad i=1,...,N
\]

one for each instance of the random uncertainty. We are now ready to formulate the scenario optimization (27) for the problem under study. In this work the optimization parameter vector \(\theta_{so}\) includes all the free parameters in the MILs (17), (19), (20), (22) and (35), that is \(\theta_{so}=[P, P, P, \Gamma, \beta, \rho_1, \rho_2, ..., \rho_n, \bar{K}, \eta]\). The vector of the constraints is named as \(f_{so}(\theta_{so}, q)\) and collects the LMI s (17), (19), (20), (22) plus the \(N\) scenario MILs defined in (38). The resulting convex optimization problem is

\[
\text{(SO): minimize } J(\theta_{so}) \text{ s.t. } f_{so}(\theta_{so}, q(i)) \quad i=1,...,N
\]

**Remark-5:** An important aspect of the stochastic approach is that the stochastic LMI s (35) descend 'immediately' from inequality (33) without any majoration or introduction of auxiliary intermediate matrices and coefficients. On the other side the robust worst-case LMI condition (23) was derived following a series of intermediate majorations that introduce conservatism in the solution. The disadvantage of the SO approach is that the number of MIL constraints \(f_{so}(\theta_{so}, q)<0\) in (39) depends on \(N\), this implies that the number of LMI s in (39) is significantly larger than the number of MILs constraints \(f_{so}(\theta_{so}, q)<0\) involved in (25). It is underlined that the parameters in \(q=[\Delta_0, W, W, r, W]\) are assumed as random parameters only at the design level. Actually, these parameters are constant or time varying bounded quantities.

**A. Solving Procedure**

The solving procedure of section IV-B used for solving the RO problem in (25) can be easily rearranged to solve the SO problem (39).

**Remark-6:** It should be empathized that both the RO and the SO method provide sufficient conditions for the satisfaction of the performance requirements, therefore unfeasibility does not mean, in general, that the performance cannot be fulfilled. It means that the considered method is not able to guarantee the performance for the current values of the design parameters. In other words both approaches introduce some form of conservativeness.

**VII. Design example: estimation of the region of operation of an F16 aircraft**

The worst-case and the probabilistic design were applied for the estimation of the region of operation of the short period longitudinal dynamics of an F16 aircraft model under MRAC control in the presence of matched and input uncertainty. It was considered the dynamical model reported in [24], that is

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
-1.01 & 0.90 \\
0.82 & -1.07
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
-0.0022 \\
-0.1756
\end{bmatrix}
\Delta(x(t))-u_{f_0}(t)
\]

where \(x_1(t)\) is the angle of attack \(\alpha(t)\), \(x_2(t)\) is the pitch rate \(q(t)\) and \(u_{f_0}(t)\) is the actuator filtered elevator command. The actuator (8) is modelled as an uncertain first order low-pass filter \(G(s) = \lambda(1+\Delta)/s+\lambda\). The actuator gain uncertainty was assumed in the range \(\pm 20\%\) that is \(\delta_\alpha = 0.2\) and the time constant of the actuator was fixed at \(\tau = \lambda^{-1} = 0.5\) s. The matched uncertainty \(\Delta(x)\) is linearly parameterized as \(\Delta(x) = W_1 x_1 + W_2 x_2\). In this study, we fixed the values \(W_1 = 0\) and \(\Delta = -1.05\) and the bound on the norm of the uncertainty was fixed at \(W_2 = 1\). The range for the reference command is \(|r(t)| \leq r_0 = 5.5\) deg. A baseline tracking controller for \(\alpha(t)\) was designed (in nominal conditions) using the pole placement technique by placing the poles of the matrix \(A = A_n + B_n K_0\) in \(s=[-1:-1.1]\). This design pro-

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duced a linear controller $K_o=[4.6698; -0.0374]$, while $K_f = -6.82$ was selected to guarantee a reference/output gain equal to one.

As for the weight adaptation algorithm, the adaptation rate was fixed at $\Gamma=10^4$ and the sigma modification term at $\sigma=10^6$. These values were defined following a detailed parametric study that was carried out in the worst-case scenario (see the comments in the next section).

A. Formulation of the performance requirements

The reference performance polyhedron $\Pi_{ref}(\rho_f=1)$ was defined by fixing $|e_1(t)| \leq e_{M1} = 1$ and $|e_2(t)| \leq e_{M2} = 1$, while the initial condition for the tracking error was defined by taking the semi axes length of the ellipsoid $\Omega_e$ equal to $e_{10} = e_{M1}/3$ and $e_{20} = e_{M2}/3$. The Multiobjective cost function was defined as $J = \xi_1 \cdot \rho_1 + \xi_2 \cdot \rho_2 + \chi \cdot \eta - \zeta \cdot \beta_i$ with weights $\xi_1 = 10^{-3}$, $\xi_2 = 10^{-3}$, $\chi = 10$, and $\zeta = 1$. The values of the weights in $J$ were defined experimentally with the purpose of achieving, at the same time, a reduced size for $\Omega_e$ and a reduced authority (norm) for $K_c$. This cost function was used both for the worst-case and for the probabilistic design.

B. Robust worst case design

In the RO worst-case design framework we are requested to solve the convex optimization problem (25). Now, defining $\bar{K}_e = K_c^T P_i$ and considering $\beta$ as a grid optimization parameter, the problem (25) becomes linear with respect to the new subset of optimization variables $\theta_{rot} = [P, P_1, P_2, \Gamma, \alpha, \gamma, e_1, e_2, e_3, \rho_1, \rho_2, \bar{K}_e, \eta]$, that is the problem

$$\min_{\theta_{rot}} J(\theta_{rot}) \quad \text{s.t.} \quad f_{ro}(\theta_{rot}) < 0$$

(41)

is linear in the optimization variables, thus implying that the minimization can be carried out applying the solving procedure proposed in section VI.B. The grid search over $\beta$ was performed in the range $\beta=[0.0001-0.03]$ with step 0.001. The optimized scaling coefficients $\rho_1$ and $\rho_2$ and the scaled norm $\|K_e\|$ as function of $\beta$ are shown in figure 1. The minimum for $J$ was reached for $\beta=0.018$ where $\rho_1=2.98$ and $\rho_2=3.25$ and the optimal gain matrix is $K_e = [13.54; -15.31]$. It is observed that for $\beta=0.018$ also the control authority (norm of $K_c$) is almost minimal. This proves that the multiobjective cost function $J$ was tuned correctly. The value $\beta=0.018$ was then fixed and used for the subsequent analysis.

Further parametric studies were carried out to evaluate the effects of the design parameters on the performance. In particular it was investigated the role of the adaptation rate $\Gamma$. It was observed that an increase in $\Gamma$ implies a decrease in the linear control authority $\|K_e\|$. We also observed that for $\Gamma>300$ there is no significant impact on the size of the invariant set $\Omega_e$. Instead, we noticed in simulation that higher adaptation rates are needed in order to improve the tracking error performance. Therefore, in the present study, the high learning rate is not primarily used to influence the size of the region $\Omega_e$ but just to improve the tracking error.
C. Probabilistic design

In the PO design framework we are requested to solve the convex optimization problem (39). In this study the parameter $\beta$ was fixed to the optimized value archived in the worst-case design that is $\beta=0.018$. The set of optimization variables in (39) is $\theta_{opt}=[P, P_1, P_2, \Gamma, \rho_1, \rho_2, K, \eta]$ while the number of independent optimization parameters in $\theta_{opt}$ is $n_0=13$. The $N$ instances (scenarios) of the stochastic variables $q=[\Delta_n, W, W, r, \tilde{W}]$ in the probabilistic LMI $\mathcal{Z}(\Delta_n, W, W, r, \tilde{W}, r)<0$ in (38) were generated according to the distributions defined in (35) and (36). The number of scenarios $N$ was computed assuming an accuracy $\varepsilon=0.05$ and a confidence level $\delta=0.0001$. Applying (30) the number of scenarios resulted $N=1180$.

The convex optimization (39) was then executed producing the following optimized values for the scaling coefficients: $\rho_1=1.20$ and $\rho_2=1.31$ while the optimal gain matrix was $K=[3.08; 2.22]$.

Figure 2 shows the comparison between the optimized performance ellipsoid $\Omega_{e(\text{ro})}$ achieved by RO design and the performance ellipsoid $\Omega_{e(\text{po})}$ achieved with the PO design. It is evident that the PO design produces a significantly smaller invariant region $\Omega_e$ compared to the RO worst-case design for the same set of uncertainties.

It can be also observed that $\Omega_{e(\text{ro})}$ is much larger than the initial condition set $\Omega_0$. This puts evidence in fact that the RO design introduces a significant conservatism in the design compared to PO design. In figure 2 are also shown some closed loop trajectories starting from the boundary of the $\Omega_0$ set. In can be observed that some trajectories evolve for some time outside $\Omega_0$ but, correctly, remains within the invariant sets $\Omega_{e(\text{ro})}$ and $\Omega_{e(\text{po})}$. It is evident that the invariant set $\Omega_{e(\text{ro})}$ is a better estimate of the region of operation of the adaptive controller than $\Omega_{e(\text{po})}$ for initial condition in the set $\Omega_0$.

Figure 3 shows (for the SO solution) the tracking error performance for the angle of attach $\alpha(t)$ ($x_1(t)$) for initial conditions starting on the boundary of $\Omega_0$. In the figure it is also shown the performance of $x_1(t)$ when the adaptive control is disabled. It is evident that in almost 30 seconds the adaptive controller is able to recover an acceptable reference tracking while the disabling of the adaptive controller causes a significant degradation of the performance. Figure 4 shows the adaptive control contributions corresponding to the responses in figure 3.
Figure 2. Comparison between robust and probabilistic design. The $\Omega_0$ set represents the initial condition set for the tracking error. $\Pi_{ep(\text{ro})}$ is the optimized performance box achieved in the context of robust-worst case optimization design, while $\Omega_{e(\text{ro})}$ is the corresponding optimized invariant ellipsoidal set. The $\Pi_{ep(\text{so})}$ set is the optimized performance box achieved in the context of scenario probabilistic design, while $\Omega_{e(\text{so})}$ is the corresponding optimized invariant ellipsoidal set. In the figure are also shown some closed loop trajectory starring on the boundary of $\Omega_0$. 

Figure 3. Tracking performance for the angle of attack $\alpha(t)$ ($x_1(t)$) for initial conditions starting on the boundary of $\Omega_0$ (for the SO solution) The thick line represents the reference signal for $\alpha(t)$, while the dashed line represents the response with the adaptive control disabled.

Figure 4. Adaptive control contribution $u_{ad}(t)$ for initial conditions starting on the boundary of $\Omega_0$. 

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D. A-Posteriori analysis of the probabilistic design

In the previous section we have shown the superiority of the PO design compared to the RO design in estimating the tracking error region of operation under adaptive control. Since the PO design provides probabilistic results it is also important to perform some additional studies to evaluate the reliability of this design. In this spirit we carried out the deterministic and stochastic analysis suggested in\textsuperscript{17} to evaluate, a-posteriori, the reliability of the Scenario RO design.

Deterministic analyses

Given the optimized solution produced by the SO design, the deterministic, or worst-case, analysis (of the probabilistic design) consists of computing the so called radius of deterministic performance $\kappa_{wc}$, which is the largest value of the scalar $\kappa \in [0, 1]$ for which the constraint $\mathbb{I}(\Delta_{w}, W_{w}, r_{w}, W_{r}) < 0$ in (38) is robustly satisfied for all the uncertainties in the scaled set $Q_{k} \equiv \kappa Q$. In practice, $\kappa$ is a reduction coefficient that is used to scale (reduce) the amplitude of variation for all the stochastic parameters. Then, once this worst-case analysis is completed, the objective is to perform a probabilistic analysis beyond the radius of deterministic performance, that is, for values of $\rho > \kappa_{wc}$.

As for the probabilistic analyses we can see that the LMI $\mathbb{I}(\Delta_{w}, W_{w}, r_{w}, W_{r}) < 0$ depends multiaffinely on all the stochastic parameters. In this case, for a fixed $\kappa$, to detect the performance for all the uncertainties in $Q_{k}$ it is well-known that it suffices to check the simultaneous satisfaction of the uncertain constraint at the vertices of the hyper-rectangle $Q_{k}$.

The radius of deterministic performance was then computed for decreasing values of $\kappa$ starting from $\kappa(=1)$ with step 0.01 and checking the satisfaction of the constraints on the vertices of the hyper-rectangle $Q_{k}$, achieving the values $\kappa_{wc}(=0.71)$. This means that the probabilistic solution provided by the SO guarantees robust performance (that it is satisfied on all the vertices) if we accept a reduction of the $71\%$ for amplitude of all the uncertain parameters.

Probabilistic analyses

In this case, the a-posteriori analysis consists of designing a Monte Carlo experiment which is based on random extractions of uncertainty samples. The result of the experiment is to return an estimated probability of satisfaction of the stochastic constraints under study. Formally we are interested in providing an estimation $\hat{R}(\theta_{so})$ of the probability of perforce defined as

$$R(\theta_{so}) = \Pr(q \in Q : f(\theta_{so}, q) < 0)$$

(42)

using $N$ iid samples of the uncertain parameter vector $q$. The empirical estimation $\hat{R}(\theta_{so})$ was built by computing the ratio between the number of times the constraint $f(\theta_{so}, q(i)) < 0$ is fulfilled and the total number of $N$ samples. The sample complexity $N$ is related to the desired accuracy $\varepsilon$ and confidence $\delta$ level and can be determined using the so-called Chernoff bound\textsuperscript{39}. More precisely, given $\varepsilon$ and $\delta$, if

$$N \geq N_{chern} = \frac{1}{2 \varepsilon^{2}} \ln \frac{2}{\delta}$$

(43)

then, the inequality $|R(\theta_{so}) - \hat{R}(\theta_{so})| \leq \varepsilon$ holds with probability at least $1-\delta$. Then, the next step is to construct the so called probability degradation function, that is the plot of the probability of constraint satisfaction as a function of the radius $\kappa$. This plot is compared with the radius of deterministic performance $\kappa_{wc}$ to provide additional information about the performance degradation beyond the worst-case performance margin.

More precisely, given the SO solution, for $\kappa > \kappa_{wc}$ we computed the empirical estimate $\hat{R}(\theta_{so})$. It is important to put into evidence that this analysis can be carried out using smaller values of $\varepsilon$ and $\delta$ than those employed in the design phase. In our a-posteriori study we fixed $\varepsilon = 0.005$ and $\delta = 10^{-6}$, that provide, through the Chernoff bound (43), a sample complexity $N = 290174$.

Then, we estimated the probability degradation function in the range $\kappa = [0.71, 2.0]$ with step 0.01. For each grid point the estimated probability of performance is estimated by computing the empirical probability of bound violation. For each value of $\kappa$, the accuracy of this estimate satisfies the inequality $|R(\theta_{so}) - \hat{R}(\theta_{so})| \leq \varepsilon$ with probability at least $1-\delta$. The results are shown in figure 5 where it is reported the estimated probability as function of the deterministic radius $\kappa$. It is observed from the plot that in case a $0.8\%$ loss of probabilistic performance may be tolerated, then, the performance margin may be increased up to the original value $\kappa=1$. This implies that the SO solution satisfies the requirements with a probability of the 99.9999% for the $(99.2\pm 0.5)\%$ of all the considered uncertainties considered in the RO framework. Such an analysis provides a useful indication about the reliability of the SO method.
VIII. Conclusions

In this paper two constructive approaches for the analysis and the design of tracking error UUB regions for a MRAC system in the presence of matched uncertainty and unmodelled input dynamics due to a partially known actuator were introduced and compared. The adoption of a parameter projection strategy (that ensures the boundedness of the adaptation parameters), allowed us to consider the closed loop MRAC system as a stable linear system subject to bounded uncertainties. This allowed us to set-up a LMI-based tests to build ellipsoidal UUB regions for the error trajectories that can be computed numerically by using convex optimization tools.

In this study we compared two approaches to face the convex optimization problem in the presence of uncertainty. The first approach relies on the robust (worst-case design) control philosophy that is it guarantee the performance satisfaction for all the allowed (bounded) uncertainties. This has led to a deterministic convex optimization problem. The second approach relies on a (novel in this context) probabilistic framework where the bounded uncertainties and the bounded adaptive control are considered, at the design stage, as random variables characterized by specific probability density functions. The so called Scenario Approach was then employed to perform the convex optimization in the stochastic context.

The two approaches where then applied to the analysis and design of a MRAC controller of the short period longitudinal dynamics of an F16 aircraft model. The results of the design reveal, clearly, that the probabilistic design is able to estimate more accurately the region of operation of the tracking error compared to the invariant region provided by the robust worst-case design.

The reason of this difference is mainly due to the unavoidable matrix majoration that are requested in the derivation of the worst-case LMI conditions that should hold for all the possible uncertainties. This entailed also the introduction of a number of auxiliary optimization variables in the robust LMIs conditions that makes the optimization problem more involved.

On the other hand the adoption of the stochastic framework allowed us to consider both the uncertain parameters and the bounded adaptive control as random variables with the important advantage that stochastic LMI conditions can be immediately achieved without the need of matrix majorations and the need of auxiliary variables.

Based on the theoretical considerations and on the experimental results, we may conclude that the probabilistic approach is very promising for the estimation of non-conservative regions of operation for adaptive control systems.
Acknowledgments
Support for the second, third and fourth author was provided by the National Aeronautics and Space Administration under Grants NNX15AM51A and NNX15AN04A.

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