Application of a Robust Adaptive Control Architecture to a Spacecraft with Flexible Dynamics

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In this paper, we design a robust adaptive controller for a flexible spacecraft model. Specifically, the proposed framework involves a new and novel controller architecture involving a modification term in the update law that minimizes an error criterion involving the distance between the weighted regressor vector and the weighted system error states. This modification term allows for fast adaptation without hindering system robustness. In particular, the governing tracking closed-loop system error equation approximates a Hurwitz linear time-invariant dynamical system with $L_\infty$ input-output signals with the proposed modification term. This key feature of our framework allows for robust stability analysis of the proposed adaptive control law using $L_1$ system theory. We further show that by properly choosing the design parameters in the modification term we can guarantee a desired bandwidth of the adaptive controller, guaranteed transient closed-loop performance, and an a priori characterization of the size of the ultimate bound of the closed-loop system trajectories. A numerical illustrative study is provided to demonstrate the efficacy of the proposed design for a flexible spacecraft model in the presence of system uncertainties and exogenous disturbances.

I. Introduction

Although adaptive control has been used in numerous applications to achieve system performance without excessive reliance on system models, the necessity of high-gain feedback for achieving fast adaptation can be a serious limitation of adaptive controllers\(^1\). Specifically, in certain aerospace applications, such as high performance aircraft systems, fast adaptation is required to achieve stringent tracking performance specifications in the face of large system uncertainties and abrupt changes in system dynamics. However, fast adaptation achieved through high-gain feedback can result in high-frequency oscillations, which can excite unmodeled system dynamics resulting in system instability\(^1,2\). Hence, there exists a critical trade-off between system stability and control adaptation rate.

Most adaptive control methods developed in the literature have averted the problem of high-gain control. Notable exceptions include Refs. 3–5. Specifically, the authors in Ref. 3 use a low-pass filter that effectively subverts high frequency oscillations that can occur because of fast adaptation, whereas using a predictor model to reconstruct the reference system model. In particular, the authors in Ref. 3 develop a robust
adaptive control architecture that provides sufficient conditions for stability and performance in terms of $\mathcal{L}_1$-norms of the underlying system transfer functions despite fast adaptation, leading to uniform bounds on the $\mathcal{L}_\infty$-norms of the system input-output signals. In Ref. 4, an indirect adaptive control architecture is developed using a least-squares parameter estimation scheme to adjust the nominal controller parameters, thereby reducing system modeling errors and effectively utilizing a direct adaptive control law with a slower learning rate. More recently, the authors in Ref. 5 present a model reference adaptive control architecture for fast adaptation predicated on an optimal control problem. In particular, a fast adaptation algorithm is developed using the minimization of the system tracking error to derive a direct adaptive control law.

In a recent paper\textsuperscript{6}, a new adaptive control architecture for nonlinear dynamical systems is developed to address the problem of high-gain adaptive control. Specifically, the proposed framework involves a new and novel controller architecture involving a modification term in the update law that minimizes an error criterion involving the distance between the weighted regressor vector and the weighted system error states. This modification term allows for fast adaptation without hindering system robustness. In particular, the governing tracking closed-loop system error equation approximates a Hurwitz linear time-invariant dynamical system with $\mathcal{L}_\infty$ input-output signals with the proposed modification term. This key feature of our framework allows for robust stability analysis of the proposed adaptive control law using $\mathcal{L}_1$ system theory. We further show that by properly choosing the design parameters in the modification term we can guarantee a desired bandwidth of the adaptive controller, guaranteed transient closed-loop performance, and an a priori characterization of the size of the ultimate bound of the closed-loop system trajectories. In this paper, we provide a numerical study to demonstrate the efficacy of the design developed in Ref. 6 for a flexible spacecraft model\textsuperscript{7,8} in the presence of system uncertainties and exogenous disturbances.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{Z}_+$ denotes the set of nonnegative integers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^T$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $(\cdot)^+$ denotes the Moore-Penrose generalized inverse, $\| \cdot \|_2$ denotes the Euclidian norm, $a \times b$ denotes the cross product of vectors $a$ and $b$, and $\| \cdot \|_F$ denotes the Frobenius matrix norm. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix $M$, $\sigma_{\min}(M)$ (resp., $\sigma_{\max}(M)$) for the minimum (resp., maximum) singular value of the Hermitian matrix $M$, $\text{spec}(\cdot)$ for the spectrum of a square matrix, $\text{tr}(\cdot)$ for the trace operator, $\text{vec}(\cdot)$ for the column stacking operator, and $(\cdot)'$ for the Fréchet derivative.

II. Robust Adaptive Control Architecture

In this section, we give an overview of the adaptive control architecture developed in Ref. 6. We first consider system uncertainty that is perfectly parameterized for all $x \in \mathbb{R}^n$. Then, we consider uncertainty characterizations approximated over a compact subset $\mathcal{D}_x$ of $\mathbb{R}^n$ for the case when the system uncertainty cannot be perfectly parameterized and/or the system includes bounded exogenous disturbances.

Consider the nonlinear uncertain dynamical system given by

$$
\dot{x}(t) = Ax(t) + B\Delta(x(t)) + Bu(t), \quad x(0) = x_0, \quad t \geq 0,
$$

where $x(t) \in \mathbb{R}^n$, $t \geq 0$, is the state vector, $u(t) \in \mathbb{R}^m$, $t \geq 0$, is the control input, $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known matrices such that the pair $(A, B)$ is controllable, and $\Delta : \mathbb{R}^n \to \mathbb{R}^m$ is a matched system uncertainty. Furthermore, we assume that the full state is available for feedback and the control input $u(\cdot)$ is restricted to the class of admissible controls consisting of measurable functions such that $u(t) \in \mathbb{R}^m$, $t \geq 0$. 

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In addition, consider the reference system given by

\[ \dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad x_m(0) = x_0, \quad t \geq 0, \]

where \( x_m(t) \in \mathbb{R}^n, \ t \geq 0, \) is the reference state vector, \( r(t) \in \mathbb{R}^r, \ t \geq 0, \) is a bounded piecewise continuous reference input, \( A_m \in \mathbb{R}^{n \times n} \) is Hurwitz, and \( B_m \in \mathbb{R}^{n \times r}. \)

**Assumption 1.** The matched uncertainty in (1) is perfectly parameterized as

\[ \Delta(x) = W^T \beta(x), \ x \in \mathbb{R}^n, \]

where \( W \in \mathbb{R}^{s \times m} \) is an unknown constant weighting matrix satisfying \( \|W\|_F \leq w, \ w \in \mathbb{R} \) is a known positive constant, and \( \beta : \mathbb{R}^n \to \mathbb{R}^s \) is a basis function of the form \( \beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)] \) satisfying \( \|\beta(x)\|_2 \leq l_{\beta_0} + l_{\beta_1} \|x\|_2 \), where \( l_{\beta_0} \in \mathbb{R} \) and \( l_{\beta_1} \in \mathbb{R} \) are positive constants.

**Remark 1.** In general, application of standard adaptive controllers to nonlinear uncertain dynamical systems does not assume the knowledge of an upper bound \( w \) of the unknown constant weighting matrix \( W \).

That is, standard adaptive controllers can conceptually deal with uncertainties having (unrealistically) large magnitudes of \( W \) for the case where the uncertainty is perfectly parameterized. However, these controllers do not have any robustness properties, especially when the adaptation gain is chosen to be large to satisfy given performance specifications. This is due to the fact that high-gain adaptive controllers involve high-bandwidths that can result in high-frequency oscillations in the closed-loop system response and excite unmodeled dynamics resulting in system instability. In contrast, the design of robust adaptive controllers with large adaptation gains requires the knowledge of a conservative upper bound \( w \) of the unknown constant weight \( W \) (see, for example, Ref. 3).

**Remark 2.** In Assumption 1, we assume that \( \|\beta(x)\|_2 \leq l_{\beta_0} + l_{\beta_1} \|x\|_2 \) for all \( x \in \mathbb{R}^n \), which only addresses a special class of uncertainties involving a linear growth bound. Later in this section, this assumption is relaxed by addressing uncertainty parameterizations approximated over a compact set \( \mathcal{D}_x \) of \( \mathbb{R}^n \), similar to what is often done in the neural network literature. In this case, we address a larger class of system uncertainties.

Next, consider the feedback control law given by

\[ u(t) = u_n(t) + u_{ad}(t), \quad t \geq 0, \]

where the nominal control law \( u_n(t), \ t \geq 0, \) is given by

\[ u_n(t) = -K_1 x(t) + K_2 r(t), \quad t \geq 0, \]

and the adaptive control law \( u_{ad}(t), \ t \geq 0, \) is given by

\[ u_{ad}(t) = -\hat{W}^T(t) \beta(x(t)), \quad t \geq 0, \]

where \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times r} \) are nominal control gains, and \( \hat{W}(t) \in \mathbb{R}^{s \times m}, \ t \geq 0, \) is an estimate of \( W \) satisfying

\[ \dot{\hat{W}}(t) = \Gamma \text{Proj}_m \left[ \hat{W}(t), \beta(x(t)) e^T(t) P B + \gamma \Psi(\beta(x(t)), \hat{W}(t)) \right], \quad \hat{W}(0) = \hat{W}_0, \quad t \geq 0, \]

where \( \Gamma \in \mathbb{R}^{s \times s} \) is a positive-definite adaptation gain matrix, \( e(t) \triangleq x(t) - x_m(t), \ t \geq 0, \) is the system error.

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state, $P \in \mathbb{R}^{n \times n}$ is a positive-definite solution of the Lyapunov equation

$$0 = A_m^T P + PA_m + R,$$  

where $R \in \mathbb{R}^{n \times n}$ is a positive-definite matrix, $\gamma > 0$ is a scalar adaptation gain, $\Psi : \mathbb{R}^s \times \mathbb{R}^{s \times m} \to \mathbb{R}^{s \times m}$ is given by

$$\Psi(\beta(x(t)), \dot{W}(t)) \triangleq -\beta(x(t)) \beta^T(x(t)) \dot{W}(t)B^T \gamma \dot{M} \dot{W}, \quad t \geq 0,$$  

where $M \in \mathbb{R}^{n \times n}$ is a positive-definite matrix chosen such that $\det[\gamma \dot{M}] \neq 0$ and $B^T \gamma \dot{M}^{-1} B$ is positive-definite, and $\hat{W}_{\max} \in \mathbb{R}$ is a norm bound imposed on the update weight $\dot{W}(t)$, $t \geq 0$. Note that $\text{Proj}_m(\cdot)$ in (7) denotes the projection operator defined in Ref. 6.

For the nominal control law (5) we have the following assumption involving standard matching conditions for model reference adaptive control.

**Assumption 2.** There exist gain matrices $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times r}$ such that $A_m = A - BK_1$ and $B_m = BK_2$.

**Remark 3.** For system uncertainty suppression, a standard adaptive controller has the form given by

$$\dot{W}(t) = \Gamma \text{Proj}_m \left[ \dot{W}(t), \beta(x(t))e^T(t)PB \right], \quad \dot{W}(0) = \dot{W}_0, \quad t \geq 0.$$  

The proposed weight update law given by (7) can be viewed as a generalization of the optimal control modification approach to the adaptive controller proposed in Ref. 5. Specifically, if $M = A_m P^{-1} - \gamma P^{-1} A_m^T$, then (7) specializes to the weight update law given in Ref. 5. However, in this case, $M$ need not be positive definite. As we see in the next sections, by properly choosing $M$ we can guarantee a desired bandwidth of the adaptive controller, guaranteed transient closed-loop performance, and we can guarantee the size of the ultimate bound of the closed-loop system trajectories.

### III. Stability and Performance Guarantees

To analyze the stability properties of the proposed robust adaptive control architecture, we write the nonlinear uncertain dynamical system (1) as

$$\dot{x}(t) = A_m x(t) + B_m r(t) - B \dot{W}^T(t) \beta(x(t)), \quad x(0) = x_0, \quad t \geq 0,$$  

where $\dot{W}(t) \triangleq \dot{W}(t) - \dot{W}, \quad t \geq 0$, with the system error and weight update error dynamics given by, respectively,

$$\dot{e}(t) = A_m e(t) - B \dot{W}^T(t) \beta(x(t)), \quad e(0) = 0, \quad t \geq 0,$$  

$$\dot{\dot{W}}(t) = \dot{\dot{W}}(t), \quad \dot{\dot{W}}(0) = \dot{W}_0.$$  

We note here that since the weight update law for $\dot{W}(t)$, $t \geq 0$, is predicated on the projection operator and $\dot{W}$ is a constant matrix, there exists a norm bound $\dot{w}_{\max}$ such that $\|\dot{W}(t)\|_F \leq \dot{w}_{\max}, \quad t \geq 0$.

**Theorem 1** (Ref. 6). Consider the nonlinear uncertain dynamical system given by (1) with reference system given by (2), and assume that Assumptions 1 and 2 hold. Furthermore, let the adaptive control law be given by (6) with weight update law given by (7). Then, the closed-loop error signals given by (12) and (13) are uniformly bounded for all $(e(0), \dot{W}(0)) \in D_0$, where $D_0$ is a compact positively invariant set, with
ultimate bound $\|e(t)\|_2 < \varepsilon$, $t \geq T$, where

$$\varepsilon \triangleq \left[ \lambda_{\text{max}}(P) \vartheta^2 + \lambda_{\text{max}}(T^{-1}) w_{\text{max}}^2 \right]^{\frac{1}{2}},$$

$$\vartheta \triangleq \left[ \frac{c_2^2}{4c_1} + \frac{c_3}{2c_1} \right] \hat{\vartheta},$$

$c_1 \triangleq \lambda_{\text{min}}(R) - \frac{1}{2} \gamma \|S\|_F^2 \gamma w_{\beta_1}^2 > 0$, $c_2 \triangleq \gamma \|S\|_F^2 \gamma w_{\beta_1}^2 l_{\beta_1}$, $c_3 \triangleq \frac{1}{2} \gamma \|S\|_F^2 \gamma w_{\beta_1}^2 l_{\beta_0} > 0$, $S \triangleq B^T \left[ \gamma A_m P^{-1} + m \right] B$, $l_{\beta_0} \triangleq l_{\beta_0} + l_{\beta_1} x_m$, and $|x_m(t)| \leq x_m, t \geq 0$. In addition, for $t \in [0, T)$, the system error and weight update error dynamics satisfy

$$\|e(t)\|_2 \leq \|\hat{W}(0)\|_F \left[ \|\Gamma^{-1}\|_F / \lambda_{\text{min}}(P) \right]^{\frac{1}{2}},$$

$$\|\hat{W}(t)\|_F \leq \|\hat{W}(0)\|_F \left[ \|\Gamma^{-1}\|_F / \lambda_{\text{min}}(\Gamma^{-1}) \right]^{\frac{1}{2}}.$$  

**Remark 4.** In Theorem 1 we assume that $\lambda_{\text{min}}(R) - \frac{1}{2} \gamma \|S\|_F^2 \gamma w_{\beta_1}^2 > 0$. Since a conservative upper bound $w$ of the unknown constant weight $W$ is known, this can be easily satisfied by judiciously choosing the design parameters. As noted in Remark 1, similar conditions are required to be satisfied in the design process for guaranteeing a robust adaptive controller design (see, for example, Ref. 3).

**Remark 5.** Theorem 1 shows that over a transient finite-time $T$ the closed-loop error signals (12) and (13) are bounded from above by (16) and (17), respectively. This implies along with uniform ultimate boundedness of the closed-loop error signals (12) and (13) that $e(\cdot) \in L_\infty$ and $\vec{W}(\cdot) \in L_\infty$, and hence, $x(\cdot) \in L_\infty$ and $u(\cdot) \in L_\infty$. Furthermore, note that $e(t), t \in [0, T)$, can be made sufficiently small (satisfying $\hat{\vartheta}$ in (15)) by letting $\lambda_{\text{min}}(\Gamma) \rightarrow \infty$.

**Remark 6.** Let $M = m I$ in (9). Since $\gamma \left[ \gamma A_m P^{-1} + m I \right]^{-1} = [A_m P^{-1} + \gamma^{-1} m I]^{-1}$, it follows that $[A_m P^{-1} + \gamma^{-1} m I]^{-1} = 0$ as $\gamma \rightarrow 0$ or $m \rightarrow \infty$. That is, $\gamma \Psi(\beta(x(t)), \hat{W}(t)) = 0$ as $\gamma \rightarrow 0$ or $m \rightarrow \infty$. In this case, we obtain the standard adaptive controller given by (10). Furthermore, it follows from (15) that $\hat{\vartheta} = 0$, that is, we recover asymptotic stability of the system error dynamics. However, $\gamma \rightarrow 0$ or $m \rightarrow \infty$ (resp., $\lambda_{\text{min}}(M) \rightarrow \infty$) are not desirable design choices since they might lead to a high bandwidth adaptive controller that can result in high-frequency oscillations in the closed-loop system response and excite unmodeled dynamics resulting in system instability. This is discussed later below.

Next, we relax Assumption 1 to the case where $\|\beta(x)\|_2 \leq l_{\beta_0} + l_{\beta_1} \|x\|_2$ does not hold for all $x \in \mathbb{R}^n$ and the system uncertainty $\Delta(x)$ cannot be perfectly parameterized.

**Assumption 3.** The matched uncertainty in (1) is linearly parameterized as

$$\Delta(x) = W^T \beta(x) + \eta(x), \quad x \in \mathcal{D}_x,$$

where $W \in \mathbb{R}^{n \times m}$ is an unknown constant weighting matrix satisfying $\|W\|_F \leq w$, $w \in \mathbb{R}$ is a known positive constant, $\beta : \mathcal{D}_x \rightarrow \mathbb{R}^n$ is a basis function of the form $\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]$ satisfying $\|\beta(x)\|_2 \leq l_{\beta_0} + l_{\beta_1} \|x\|_2$, $x \in \mathcal{D}_x$, where $l_{\beta_0} \in \mathbb{R}$ and $l_{\beta_1} \in \mathbb{R}$ are positive constants, $\eta : \mathcal{D}_x \rightarrow \mathbb{R}^m$ is the system modeling error satisfying $\|\eta(x)\|_2 \leq \eta_{\text{max}}$, $\eta_{\text{max}} > 0$.

**Remark 7.** Note that Assumption 3 does not assume boundedness of the basis function $\beta(x), x \in \mathcal{D}_x$, as is standard in the neuroadaptive control literature. Furthermore, in this case, a basis function constructed using neural networks is not the only choice for $\beta(x)$ since the boundedness assumption on $\beta(x), x \in \mathcal{D}_x$, that is, $\|\beta(x)\|_2 \leq l_{\beta_0}, x \in \mathcal{D}_x$, is relaxed to $\|\beta(x)\|_2 \leq l_{\beta_0} + l_{\beta_1} \|x\|_2, x \in \mathcal{D}_x$.

Using the uncertainty parametrization given by (18), the system error dynamics are now given by

$$\dot{e}(t) = A_m e(t) - B W^T(t) \beta(x(t)) + B \eta(x(t)), \quad e(0) = 0, \quad t \geq 0,$$

where $\delta(t)$ is the control input.
where $\tilde{W}(t), t \geq 0$, is given by (13).

**Theorem 2** (Ref. 6). Consider the nonlinear uncertain dynamical system given by (1) with reference system given by (2), and assume that Assumptions 2 and 3 hold. Furthermore, let the adaptive control law be given by (6) with weight update law given by (7). Then, the closed-loop error signals given by (19) and the system given by (2), and assume that Assumptions 2 and 3 hold. Furthermore, let the adaptive control law of Theorem 2 are satisfied. In the case where $\Delta(\cdot)$ is continuous on $\mathbb{R}^n$, it follows from the Stone-Weierstrass theorem that $\Delta(\cdot)$ can be approximated over an arbitrarily large compact set $D_x$ in the sense of (18).

Next, we show that the governing tracking closed-loop system error equation approximates a Hurwitz linear time-invariant dynamical system with $L_\infty$ input-output signals. This allows us to derive uniform transient and steady-state performance bounds in terms of $L_1$-norms of the closed-loop system error dynamics that are independent of the system adaptation rate. The following lemma is key in developing the results of this section.

**Lemma 1** (Ref. 6). Consider the weight update law given by (7). If $M = -\gamma A_m P^{-1} + m I, m > 0$, then

$$B \tilde{W}(t) \beta(x(t)) = \gamma^{-1} m P_B P e(t) - \gamma^{-1} m B^T \left[ \beta^T(x(t)) \left[ \Gamma^{-1} \tilde{W}(t) - \kappa(t) \right] \right]^T, \quad t \geq 0,$$

where $P_B \triangleq B (B^T B)^+ B^T, B^\perp \triangleq (B^T B)^+ B^T, \beta^T(x(t)) \triangleq \left( \beta^T(x(t)) \beta(x(t)) \right)^+ \beta(x(t))^T$, and $\kappa(t) \in \mathbb{R}^{s \times m}$, $t \geq 0$, is a residual error signal.

**Remark 9.** The residual error $\kappa(t) \in \mathbb{R}^{s \times m}, t \geq 0$, in (22) satisfies $\|\kappa(t)\|_F \leq \kappa_{\max}, t \geq 0$, since all the signals in (22) are bounded by Theorem 1 (resp., Theorem 2). Furthermore, $\kappa(t), t \geq 0$, is identically zero when $\tilde{W}_{ij}(t) \in (-\tilde{w}_{\max}, \tilde{w}_{\max})$, for all $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, m$, where $\tilde{w}_{\max}$ is the projection bound of $\tilde{W}(t), t \geq 0$. Hence, by choosing a conservative projection bound $\tilde{w}_{\max}, \kappa(t) = 0$ for all $t \geq 0$.

**Proposition 1** (Ref. 6). Consider the system error dynamics given by (12) (resp., (19)) and let $M = -\gamma A_m P^{-1} + m I, m > 0$. Furthermore, assume Assumption 1 (resp., Assumption 3) holds. Then, the system error dynamics (12) (resp., (19)) can be characterized by the linear, time-invariant system $\mathcal{G}_{e, w}$ given by

$$\dot{e}(t) = \tilde{A} e(t) + \tilde{B} w(t), \quad e(0) = e_0, \quad t \geq 0,$$

$$z(t) = \tilde{C} e(t),$$

where $\tilde{A} \triangleq A_m - \gamma^{-1} m P_B P$ is Hurwitz, $\tilde{B} \triangleq B$, $\tilde{C} \triangleq I$, $w(t) \triangleq \gamma^{-1} m (B^T B)^+ \left[ \beta^T(x(t)) \left[ \Gamma^{-1} \tilde{W}(t) - \kappa(t) \right] \right]^T + W^T \beta(x(t)) + \eta(x(t))$, $\eta(x(t))$ is a residual error signal.

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and \( w(\cdot) \in \mathcal{L}_\infty \).

Note that (23) and (24) characterizes the closed-loop linear time-invariant dynamical system. Since \( w(\cdot) \in \mathcal{L}_\infty \) and \( \hat{A} \) is Hurwitz, it follows that \( z(\cdot) \in \mathcal{L}_\infty \).

**Theorem 3** (Ref. 6). Consider the linear, time-invariant system \( \mathcal{G}_{e,w} \) given by (23) and (24). Let \( \alpha > 0 \) be such that \( \hat{A} + \frac{\gamma}{2} I \) is Hurwitz and let \( Q_\alpha \in \mathbb{R}^{n \times n} \) be the unique, nonnegative definite solution to the Lyapunov equation \( 0 = \hat{A}Q_\alpha + Q_\alpha \hat{A}^T + \alpha Q_\alpha + \hat{B}\hat{B}^T \). Then, \( \mathcal{G}_{e,w} : \mathcal{L}_\infty \to \mathcal{L}_\infty \) and \( \| \mathcal{G}_{e,d} \|_{(\infty,2),(\infty,2)} \leq \frac{1}{\sqrt{n}} \max_{m \in [7,\infty]}(Q_\alpha) \).

The following proposition discusses the special case when \( \lambda_{\text{min}}(I) \to \infty \) and \( \hat{W}_ij(t) \in (-\hat{w}_{\max}, \hat{w}_{\max}) \) for all \( i, j = 1, 2, \ldots, s \) and \( j = 1, 2, \ldots, m \), where \( \hat{w}_{\max} \) is the projection bound of \( \hat{W}(t) \), \( t \geq 0 \). In this case, \( \Delta(x(t)), t \geq 0 \), in (3) (resp., in (18)) is directly related to \( z(t), t \geq 0 \), in (24), through \( \mathcal{G}_{e,w} \).

**Proposition 2** (Ref. 6). Consider the system error dynamics given by (12) and let \( M = -\gamma A_m P^{-1} + mI \), \( \gamma > 0 \), and \( \Gamma = \xi \Gamma \xi, \xi > 0, \Gamma \xi > 0 \). Furthermore, assume Assumption 1 (resp. Assumption 3) holds and \( \hat{W}(t), t \geq 0 \), is bounded as \( \xi \to \infty \). If \( \xi \to \infty \), then the system error dynamics (12) (resp. (19)) can be characterized by the linear, time-invariant system \( \mathcal{G}_{e,w} \) given by (23) and (24), where \( \hat{A} \equiv A_m - \gamma^{-1} m P_B B \) is Hurwitz, \( \hat{B} \equiv B, \hat{C} \equiv I \), \( w(t) \equiv \gamma^{-1} m(B^T B)^+ [-\beta \hat{L}(x(t)) \kappa(x(t))]^T + W^T \beta(x(t)) \) (resp., \( w(t) \equiv \gamma^{-1} m(B^T B)^+ [-\beta \hat{L}(x(t)) \kappa(x(t))]^T + W^T \beta(x(t)) + \eta(x(t)) \)), and \( w(\cdot) \in \mathcal{L}_\infty \). If, in addition, \( \hat{W}_ij(t) \in (-\hat{w}_{\max}, \hat{w}_{\max}) \), for all \( i, j = 1, 2, \ldots, s \) and \( j = 1, 2, \ldots, m \), where \( \hat{w}_{\max} \) is the projection bound of \( \hat{W}(t), t \geq 0 \), then \( w(t) \equiv \Delta(x(t)) \), where \( \Delta(x(t)), t \geq 0 \), is given by (3) (resp., (18)), and \( w(\cdot) \in \mathcal{L}_\infty \).

**Remark 10.** Proposition 2 characterizes a closed-loop linear time-invariant dynamical system from the system input \( \Delta(x(t)), t \geq 0 \), given by (3) (resp., (18)), to the system output \( e(t), t \geq 0 \). Note that for both results, we assume the time derivative of \( \hat{W}(t), t \geq 0 \), remains bounded as \( \xi \to \infty \). This assumption is not restrictive since the modification term given by (9) can be viewed as a variable and weighted damping term due to its sign definiteness and dependence on \( \hat{W}(t), t \geq 0 \), and hence, it limits the frequency content of the weight update law given by (7) resulting in a bounded time derivative of \( \hat{W}(t), t \geq 0 \). This observation holds even in the face of infinitely large adaptation gains.

**Remark 11.** By choosing a conservative projection norm bound \( \hat{w}_{\max} \) (see Remark 9) and letting \( \lambda_{\text{min}}(I) \to \infty \), the closed-loop linear time-invariant dynamical system given by (23) and (24) characterizes a mapping when \( \Delta(x(t)), t \geq 0 \), given by (3) (resp., (18)) is viewed as a system input and \( e(t), t \geq 0 \), is viewed as a system output. Hence, closed-loop system properties such as system bandwidth, transient system performance, and \( L_1 \) system norm of \( \mathcal{G}_{e,w} \) (Theorem 3) can be analyzed using (23) and (24).

As noted in Remark 6, \( \gamma \to 0 \) or \( m \to \infty \) (resp., \( \lambda_{\text{min}}(M) \to \infty \)) are not always desirable design choices. This is because in this case \( \mathcal{G}_{e,w} \) has infinitely fast decaying poles in the left-half plane, and hence, \( \mathcal{G}_{e,w} \) behaves like a high-bandwidth filter, which can be very sensitive to variations in \( w(t) \) causing high-frequency oscillations in the system response. Hence, by varying the parameters \( \gamma \) and \( M \) we can a guarantee a prescribed closed-loop system bandwidth.

Finally, we show that the previous results can be easily applied to the disturbance rejection problem. To see this, replace \( \Delta(x(t)) \) in (1) with \( d(t) \in \mathbb{R}^m, t \geq 0 \), where \( d(t) \) is an unknown disturbance satisfying \( \| d(t) \|_2 \leq d_{\max}, t \geq 0 \), and \( \| d(t) \|_2 \leq \hat{d}_{\max}, t \geq 0 \). In this case, the adaptive control law in (6) becomes \( u_{ad}(t) = -\hat{d}(t), t \geq 0 \), where \( d(t) \in \mathbb{R}^m, t \geq 0 \), is an estimate of \( d(t), t \geq 0 \), satisfying \( \hat{d}(t) = \Gamma \text{Proj}[\hat{d}(t), B^T P e(t) - \gamma B \hat{L}(x(t)) \kappa(x(t))] B \hat{d}(t)], \hat{d}(0) = \hat{d}_0, t \geq 0 \), with \( M \in \mathbb{R}^{n \times n} \) being a positive-definite matrix chosen such that \( \det[\gamma A_m P^{-1} + M] \neq 0 \) and \( B^T \gamma A_m P^{-1} + M \) is positive definite, and \( \hat{d}_{\max} \in \mathbb{R} \) being a norm bound imposed on the update weight \( \hat{d}(t), t \geq 0 \). An identical result to Theorem 1 can now be established by considering the Lyapunov-like function candidate given by \( V(e, \hat{d}) = e^T P e + \hat{d}^{-1} \hat{d}, \) where \( \hat{d}(t) \equiv \hat{d}(t) - d(t), t \geq 0 \). Furthermore, if we set...
\[ M = -\gamma A_m P^{-1} + mI, \ m > 0, \ \text{then using similar arguments as in Lemma 1 it can be shown that} \]
\[ B\hat{d}(t) = \gamma^{-1}mP_B P(t) - \gamma^{-1}mB^L [\Gamma^{-1} d(t) - \kappa(t)], \ t \geq 0, \ \text{with} \kappa(t) \in \mathbb{R}^m, \ t \geq 0, \ \text{being the residual error signal. In this case, identical results to Propositions 1 and 2 and Theorem 3 can be easily obtained.} \]

**IV. Application to a Flexible Spacecraft**

In this section, we apply the proposed robust adaptive control architecture to the stabilization of a flexible spacecraft\(^7,8\). It is shown that the proposed modification term (9) results in improved system performance. Specifically, we first show that a standard adaptive architecture (10) cannot simultaneously achieve a fast convergence rate and obtain a smooth system response. Large adaptive gains lead to a highly oscillatory system response, large control inputs, and unrealistic control input rates. On the other hand, small adaptive gains result in slow convergence rates. However, when the proposed adaptive control architecture (7) is used the system is able to achieve a fast convergence rate and obtain a smooth system response, simultaneously. In addition, the maximum control input and control input rate are decreased and the flexible dynamics of the system were less excited.

Under the hypothesis of small elastic deformations\(^8\), the dynamics of a spacecraft with flexible appendages is given as

\[
J\dot{\omega}(t) + \omega(t) \times [J\omega(t) + \delta^T \eta(t)] + \delta^T \dot{\eta}(t) = u(t) + d(t), \ \ \ \omega(0) = \omega_0, \ t \geq 0,
\]

\[
\dot{\eta}(t) + C\dot{\eta}(t) + K\eta + \delta \dot{\omega}(t) = 0, \ \ \ \eta(0) = \eta_0, \ \ \ \eta(0) = \eta_0,
\]

where \( J \in \mathbb{R}^{3 \times 3} \) is the inertia matrix of the spacecraft and its flexible appendages, \( \omega \in \mathbb{R}^3 \) is the angular velocity of the spacecraft with respect to an inertial frame expressed in the body frame, \( \eta \in \mathbb{R}^N \) is the modal coordinate vector relative to the main body, \( u \in \mathbb{R}^3 \) is the control input vector, \( d \in \mathbb{R}^3 \) is a bounded external disturbance, \( \delta \in \mathbb{R}^{N \times 3} \) denotes the coupling matrix between the system’s flexible and rigid dynamics,

\[
K = \text{diag}(2\xi_1\Lambda_1, \ldots, 2\xi_N\Lambda_N) \in \mathbb{R}^{N \times N},
\]

and

\[
C = \text{diag}(\Lambda_1^2, \ldots, \Lambda_N^2) \in \mathbb{R}^{N \times N}
\]

are the damping and stiffness matrix, respectively, with \( N \) denoting the number of elastic modes considered and \( \Lambda_i \) and \( \xi_i, \ i = 1, \ldots, N \) denoting the \( i^{th} \) mode’s natural frequency and the associated damping, respectively. For all our simulations the following *unknown* system parameters are considered\(^8\)

\[
J = \begin{bmatrix}
350 & 3 & 4 \\
3 & 270 & 10 \\
4 & 10 & 190
\end{bmatrix} \text{kg} \cdot \text{m}^2,
\]

\[
\delta = \begin{bmatrix}
6.45637 & 1.27814 & 2.15629 \\
-1.25819 & 0.91756 & -1.67264 \\
1.11687 & 2.48901 & -0.83674
\end{bmatrix} \text{kg}^{1/2} \cdot \text{m/s}^2,
\]

\[
\Lambda_1 = 0.7681, \ \Lambda_2 = 1.1038, \ \Lambda_2 = 1.8733, \ \xi_1 = 0.0056, \ \xi_2 = 0.0086, \ \text{and} \ \xi_3 = 0.013. \ \text{In addition, the}
\]

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exogenous disturbance is given as the constant vector

\[ d = [0.2, 0.1, -0.1]^T. \] (31)

The unit quaternion is used to represent the spacecraft’s attitude. Specifically, the resulting kinematic equations are given by

\[ \dot{q}(t) = \frac{1}{2}(q(t) \times \omega(t) + \bar{q}(t)\omega(t)), \quad q(0) = q_0, \quad t \geq 0, \] (32)

\[ \dot{\bar{q}}(t) = \frac{1}{2}q^T(t)\omega(t), \quad \bar{q}(0) = \bar{q}_0, \] (33)

where \([\bar{q}, q]^T \in \mathbb{R}^4\) denotes the unit quaternion vector representing the attitude orientation of the spacecraft in the body framework with respect to the inertial frame. For all our simulations the initial conditions of the system are given by

\[ \omega(0) = [0.2, -0.4, -0.1]^T, \eta(0) = [1, -1, 1]^T, \bar{q}(0) = [-1, 1, 1]^T, q(0) = [-0.4, -0.2, 0.8]^T, \text{ and } \bar{q}(0) = 0.4. \]

The ideal reference model is given as

\[ \dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad x_m(0) = x_0, \quad t \geq 0, \] (34)

where

\[ A_m = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.5 \\ -1.0 & 0 & 0 & -1.4142 & 0 & 0 \\ 0 & -1.0 & 0 & 0 & -1.4142 & 0 \\ 0 & 0 & -1.0 & 0 & 0 & -1.4142 \end{bmatrix}, \] (35)

\[ B_m = [0_{3 \times 3} I_3], x_m(t) = [q^T \omega]^T \in \mathbb{R}^6, \text{ and } r(t) \in \mathbb{R}^3 \text{ denotes the reference input. Since stabilization is sought we set } r(t) \equiv 0 \text{ and } x_m(t) \equiv 0, \text{ and hence, } e(t) = x(t) \text{ in the adaptive architecture (7) and the standard adaptive architecture (10).} \]

The adaptive control architecture is used to mitigate the affects of the unknown system parameters and exogenous disturbances. Since the system parameters are considered to be completely unknown the nominal control law (5) is not utilized (i.e., \(u_n(t) \equiv 0, t \geq 0\) and only the adaptive control law (6) is used. Furthermore, we assume \(x(t) = [q(t)T \omega(t)]^T \in \mathbb{R}^6\) is available for feedback. The basis functions considered consists of a bias and sigmoid functions and are given by

\[ \beta(x) = [1, \beta_1(x_1(t)), \ldots, \beta_6(x_6(t))], \] (36)

where

\[ \beta_i(x_i(t)) = \frac{1 - e^{x_i(t)}}{1 + e^{x_i(t)}}, \quad i = 1, \ldots, 6. \] (37)

In addition we set, \(\Gamma = \bar{\Gamma} I_7\) and \(M = 5I_6\) and vary \(\bar{\Gamma}\) and \(\gamma\) are varied in order to study their affect on system performance.
A. Case 1

Figures 1 and 2 show the response of the uncontrolled system. This response is shown to give the reader a baseline understanding of the system and to show dynamic characteristics of the system’s flexible appendages. Figures 3 and 4 show the response of the system when $\bar{\Gamma} = 25$ and $\gamma = 0$. Note that even though the system is stable its settling time is large. Furthermore, as shown in Figure 5, the maximum control input exceeds 100 N-m. Finally, the peak values of $\eta(t)$, $t \geq 0$, and $\dot{\eta}(t)$, $t \geq 0$ exceed that of the uncontrolled system. Thus, the adaptive controller excites the system’s flexible dynamics further than in the uncontrolled case.

B. Case 2

Figures 6 and 7 show the response of the system when $\bar{\Gamma} = 100$ and $\gamma = 0$. Although the settling time is improved the response of the system and the control inputs are more oscillatory than in Case 1. In addition, as seen in Case 1, the maximum control input exceeds 100 N-m and the peak values of $\eta(t)$, $t \geq 0$, and $\dot{\eta}(t)$, $t \geq 0$ exceed that of the uncontrolled system.

C. Case 3

Figures 9 and 10 show the response of the system when $\bar{\Gamma} = 250$ and $\gamma = 0$. As expected, the system has a faster settling time than the previous two cases. However, all system states, including the modal coordinate vector and its derivative, are more oscillatory than in the previous two cases. In addition, the maximum control input now exceeds 200 N-m and the control input rate has increased. Furthermore, the peak values of $\eta(t)$, $t \geq 0$, and $\dot{\eta}(t)$, $t \geq 0$, exceed that of the uncontrolled system and the previous two cases. Therefore, a fast settling time and a reasonable system response cannot be obtained simultaneously with the standard adaptive control architecture.

D. Case 4

Figures 12 and 13 show the response of the system when $\bar{\Gamma} = 250$ and $\gamma = 0.05$. Note that the system has a faster settling time than Case 3. Furthermore, the system states are less oscillatory than any of the cases previously studied. In addition, the maximum control input does not exceed 30 N-m and the control input rates are lower than any of the cases previously studied. Therefore, actuator magnitude and rate saturation limits are less likely to be reached when the proposed adaptive control architecture is used. Finally, the peak values of $\eta(t)$, $t \geq 0$, and $\dot{\eta}(t)$, $t \geq 0$ are smaller than the uncontrolled system. Therefore, the proposed adaptive control architecture reduces the affect of flexible dynamics when compared to the uncontrolled system or any other case shown.

V. Conclusion

The robust adaptive control design framework developed in Ref. 6 was applied to a spacecraft with flexible dynamics. The proposed architecture goes beyond the standard adaptive control architecture by providing fast adaptation to effectively suppress system uncertainties and exogenous disturbances. The architecture is easy to implement and can be used in a complimentary way with other approaches to design robust adaptive control systems.
Figure 1. Attitude and angular velocity as functions of time of the uncontrol system.

Figure 2. Modal coordinate vector and its derivative as functions of time of the uncontrol system.
Figure 3. Attitude and angular velocity as functions of time when $\bar{\Gamma} = 25, \gamma = 0$.

Figure 4. Modal coordinate vector and its derivative as functions of time when $\bar{\Gamma} = 25, \gamma = 0$. 
Figure 5. Control input and adaptive weight time history for $\bar{\Gamma} = 25, \gamma = 0$.

Figure 6. Attitude and angular velocity as functions of time when $\bar{\Gamma} = 100, \gamma = 0$. 
Figure 7. Modal coordinate vector and its derivative as functions of time when $\bar{\Gamma} = 100, \gamma = 0$.

Figure 8. Control input and adaptive weight time history for $\bar{\Gamma} = 100, \gamma = 0$. 
Figure 9. Attitude and angular velocity as functions of time when $\bar{\Gamma} = 250, \gamma = 0$.

Figure 10. Modal coordinate vector and its derivative as functions of time when $\bar{\Gamma} = 250, \gamma = 0$. 
Figure 11. Control input and adaptive weight time history for $\bar{\Gamma} = 250, \gamma = 0$.

Figure 12. Attitude and angular velocity as functions of time when $\bar{\Gamma} = 250, \gamma = 0.05$. 
Figure 13. Modal coordinate vector and its derivative as functions of time when $\bar{\Gamma} = 250, \gamma = 0.05$.

Figure 14. Control input and adaptive weight time history for $\bar{\Gamma} = 250, \gamma = 0.05$. 

References


