On Transient Performance Improvement of Adaptive Control Architectures

Benjamin Gruenwald and Tansel Yucelen
Department of Mechanical and Aerospace Engineering
Missouri University of Science and Technology

Abstract — While adaptive control theory has been used in numerous applications to achieve given system stabilization or command following criteria without excessive reliance on mathematical models, the ability to obtain a predictable transient performance is still an important problem – especially for applications to safety-critical systems and when there is no \textit{a priori} knowledge on upper bounds of existing system uncertainties. To address this problem, we present a new approach to improve the transient performance of adaptive control architectures. In particular, our approach is predicated on a novel controller architecture, which involves added terms in the update law entitled \textit{artificial basis functions}. These terms are constructed through a gradient optimization procedure to minimize the system error between an uncertain dynamical system and a given reference model during the learning phase of an adaptive controller. We provide a detailed stability analysis of the proposed approach, discuss the practical aspects of its implementation, and illustrate its efficacy on a numerical example.

Keywords — Uncertain dynamical systems; stabilization and command following; adaptive control; transient performance improvement

* Corresponding author: 400 West, 13th Street, Rolla, MO 65409 (Address); +1 573 341 7702 (Phone); +1 573 341 6899 (Fax); yucelen@mst.edu (Email).
1. Introduction

Progress in adaptive control has been made to obtain desirable tracking and stabilization specifications while relaxing dependency on model accuracy. One of the challenges in adaptive control is to obtain a predictable transient performance [1–4]. One way to address this problem is to use a high-gain learning rate in the update law which minimizes the worst-case system error between an uncertain dynamical system and a given reference model to guarantee transient performance improvement during the learning phase. Even though this can be justified theoretically (see, for example, [5]), an update law subject to a very high learning rate is not practically feasible [6, 7], since it can lead to control signals with high-frequency dynamical system content (i.e., oscillations and high-levels of measurement noise) that can violate actuator limits [8] and excite unmodeled dynamics [9] – resulting in system instability for practical applications. The authors in [10] and [11] present high-gain adaptive controllers to subvert high-frequency dynamical system content in the control signals so that their approaches become practically feasible. Even though these approaches are promising, they require the knowledge of a (conservative) upper bound on the unknown constant gain appearing in their uncertainty parameterization. While this upper bound may be available for some applications, the actual upper bound may change and exceed its conservative estimate, for example, when an aircraft undergoes a sudden change in dynamics as a result of reconfiguration, deployment of a payload, docking, or structural damage [12]. In such circumstances, the performance of these adaptive controllers may be poor, because tuning them online with a new upper bound is not possible. Furthermore, the performance of these adaptive controllers in the face of high uncertainty levels may not be satisfactory as well, because both controllers converge to a standard adaptive controller as the upper bound on the unknown constant gain becomes arbitrarily large (see, for example, Section 2.1.2 of [10] and Section 4 of [11]). Therefore, it is important to achieve transient performance guarantees when there is no a priori knowledge on such uncertainty upper bounds.

In this paper, we present a new approach to improve the transient performance of adaptive
control architectures. In particular, our approach is predicated on a novel controller architecture, which involves added terms in the update law entitled \textit{artificial basis functions}. These terms are constructed through a gradient optimization procedure to minimize the system error between an uncertain dynamical system and a given reference model during the learning phase of an adaptive controller – without requiring \textit{a priori} knowledge on upper bounds of existing system uncertainties. The proposed approach is a theoretical and practical generalization of the method presented in [13]. Theoretically, this paper provides a stability analysis that holds for a larger class of uncertain dynamical systems. Practically, it should be noted that the method in [13] requires the differentiation of the system error, however this paper provides further results to highlight how to implement the proposed approach without this requirement, which is important for real world applications. We provide a detailed stability analysis of the proposed approach, discuss the practical aspects of its implementation, and illustrate its efficacy on a numerical example. Although this paper considers a particular adaptive control formulation, namely model reference adaptive control, the presented approach can be used in a complimentary way with many other approaches to adaptive control.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp. $\mathbb{R}_+^+$) denotes the set of positive (resp. non-negative-definite) real numbers, $\mathbb{R}^{n \times n}_+$ (resp. $\mathbb{R}_+^{n \times n}$) denotes the set of $n \times n$ positive-definite (resp. non-negative-definite) real matrices, $\mathbb{S}^{n \times n}$ denotes the set of $n \times n$ symmetric real matrices, $\mathbb{D}^{n \times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries, $(\cdot)^T$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, $\text{tr}(\cdot)$ denotes the trace operator, and $\triangleq$ denotes equality by definition. In addition we write $\lambda_{\min}(A)$ (respectively $\lambda_{\max}(A)$) for the minimum and respectively maximum eigenvalue of the Hermitian matrix $A$ and $\det(A)$ for the determinant of the Hermitian matrix $A$. We also use $\| \cdot \|_2$ for the Euclidian norm, $\| \cdot \|_\infty$ for the infinity norm, and $\| \cdot \|_F$ for the Frobenius matrix norm. Furthermore, for the signal $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathbb{R}^n$ defined for all $t \geq 0$, the truncated $\mathcal{L}_\infty$ norm and the $\mathcal{L}_\infty$ norm are defined as $\| x(t) \|_{\mathcal{L}_\infty} \triangleq \max_{1 \leq i \leq n} (\sup_{0 \leq t \leq T} | x_i(t) |)$ and $\| x(t) \|_{\mathcal{L}_\infty} \triangleq \max_{1 \leq i \leq n} (\sup_{t \geq 0} | x_i(t) |)$, respectively.
The organization of this paper is as follows. Section II considers a particular adaptive control formulation, namely model reference adaptive control, and presents the preliminaries associated with this framework. Section III introduces the proposed artificial basis function approach to model reference adaptive control and then provides performance improvement and stability results in detail. We discuss the practical aspects of the proposed approach in Section IV and present an illustrative example in Section V. Conclusions are summarized in Section VI.

2. Mathematical Preliminaries

Consider the uncertain dynamical system given by

\[ \dot{x}(t) = Ax(t) + Bu(t) + D\delta(x(t)), \quad x(0) = x_0, \quad (1) \]

where \( x(t) \in \mathbb{R}^n \) is the state vector available for feedback, \( u(t) \in \mathbb{R}^m \) is the control input, \( \delta : \mathbb{R}^n \to \mathbb{R}^m \) is an uncertainty, \( A \in \mathbb{R}^{n \times n} \) is a known system matrix, \( B \in \mathbb{R}^{n \times m} \) is an unknown control input matrix, \( D \in \mathbb{R}^{n \times m} \) is a known uncertainty input matrix, and the pair \((A, B)\) is controllable. As standard, we assume that the uncertainty in (1) can be parameterized as

\[ \delta(x) = W^T\sigma(x), \quad x \in \mathbb{R}^n, \quad (2) \]

where \( W \in \mathbb{R}^{s \times m} \) is an unknown weight matrix and \( \sigma : \mathbb{R}^n \to \mathbb{R}^s \) is a known basis function of the form \( \sigma(x) = [\sigma_1(x), \sigma_2(x), \ldots, \sigma_s(x)]^T \), and the unknown control input matrix satisfies

\[ B = D\Lambda, \quad (3) \]

where \( \det(D^TD) \neq 0 \) and \( \Lambda \in \mathbb{R}_+^{m \times m} \cap \mathbb{D}^{m \times m} \) is an unknown control effectiveness matrix.

Next, consider the reference system capturing a desired closed-loop dynamical system performance given by

\[ \dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \quad (4) \]

where \( x_r(t) \in \mathbb{R}^n \) is the reference state vector, \( c(t) \in \mathbb{R}^m \) is a given uniformly continuous bounded command, \( A_r \in \mathbb{R}^{n \times n} \) is the Hurwitz reference system matrix, and \( B_r \in \mathbb{R}^{n \times m} \) is the command
input matrix. The objective of the model reference adaptive control problem is to construct a feedback control law \( u(t) \) such that the state vector \( x(t) \) asymptotically follows the reference state vector \( x_r(t) \) subject to (2) and (3).

For the purpose of stating the preliminaries associated with the model reference adaptive control problem, consider the feedback control law given by

\[
  u(t) = u_n(t) + u_a(t),
\]

where \( u_n(t) \in \mathbb{R}^m \) is the nominal feedback control law and \( u_a(t) \in \mathbb{R}^m \) is the adaptive feedback control law. Additionally, let the nominal feedback control law be given by

\[
  u_n(t) = K_1 x(t) + K_2 c(t),
\]

where \( K_1 \in \mathbb{R}^{m \times n} \) and \( K_2 \in \mathbb{R}^{m \times m} \) are the nominal feedback and the nominal feedforward gains, respectively, such that \( A_r = A + DK_1 \), \( B_r = DK_2 \), and \( \det(K_2) \neq 0 \) holds. Now, using (5) and (6) in (1) yields

\[
  \dot{x}(t) = A_r x(t) + B_r c(t) + D \Lambda [u_a(t) + W_\sigma^T \sigma(x(t)) + W_{u_n}^T u_n(t)],
\]

where \( W_\sigma \triangleq W \Lambda^{-1} \in \mathbb{R}^{s \times m} \) and \( W_{u_n} \triangleq [I - \Lambda^{-1}] \in \mathbb{D}^{m \times m} \) are unknown.

Motivating from the structure of the uncertain terms appearing in (7), let the adaptive feedback control law be given by

\[
  u_a(t) = -\hat{W}_\sigma^T(t) \sigma(x(t)) - \hat{W}_{u_n}^T u_n(t),
\]

where \( \hat{W}_\sigma(t) \in \mathbb{R}^{s \times m} \) and \( \hat{W}_{u_n}(t) \in \mathbb{R}^{m \times m} \) are the estimates of \( W_\sigma \) and \( W_{u_n} \), respectively, satisfying the weight update laws

\[
  \dot{\hat{W}}_\sigma(t) = \gamma_\sigma \sigma(x(t)) e^T(t) PD, \quad \hat{W}_\sigma(0) = \hat{W}_{\sigma 0},
\]

\[
  \dot{\hat{W}}_{u_n}(t) = \gamma_{u_n} u_n(t) e^T(t) PD, \quad \hat{W}_{u_n}(0) = \hat{W}_{u_n 0},
\]

where \( \gamma_\sigma \in \mathbb{R}^{s \times s} \cap \mathbb{S}^{s \times s} \) and \( \gamma_{u_n} \in \mathbb{R}^{m \times m} \cap \mathbb{S}^{m \times m} \) are the learning rate matrices, \( e(t) \triangleq x(t) - x_r(t) \) is the system error state vector, and \( P \in \mathbb{R}^{n \times n} \) is a solution of the Lyapunov equation.
\[ 0 = A_r^T P + PA_r + R, \] (11)

with \( R \in \mathbb{R}^{n \times n}_+ \cap \mathbb{S}^{n \times n} \). Note that because \( A_r \) is Hurwitz, it follows from the converse Lyapunov theory [14] that there exists a unique \( P \) satisfying (11) for a given \( R \).

Now, using (8) in (7) yields

\[ \dot{x}(t) = A_t x(t) + B_t c(t) - D \Lambda [\dot{\bar{W}}_\sigma^T(t) \sigma(x(t)) + \dot{\bar{W}}_{u_n}^T(t) u_n(t)], \] (12)

and the system error dynamics is given using (4) and (12) as

\[ \dot{e}(t) = A_t e(t) - D \Lambda [\dot{\bar{W}}_\sigma^T(t) \sigma(x(t)) + \dot{\bar{W}}_{u_n}^T(t) u_n(t)], \quad e(0) = e_0, \] (13)

where \( \dot{\bar{W}}_\sigma(t) \triangleq \dot{W}_\sigma(t) - W_\sigma \in \mathbb{R}^{s \times m} \) and \( \dot{\bar{W}}_{u_n}(t) \triangleq \dot{W}_{u_n}(t) - W_{u_n} \in \mathbb{R}^{m \times m} \).

**Remark 1.** The weight update laws given by (9) and (10) can be derived using Lyapunov analysis by considering the Lyapunov function candidate (see, for example, [15])

\[ V(e, \dot{W}_\sigma, \dot{W}_{u_n}) = e^T P e + \gamma_\sigma^{-1} \text{tr} (\dot{W}_\sigma \Lambda^{1/2} (\dot{W}_\sigma \Lambda^{1/2})^\top) + \gamma_{u_n}^{-1} \text{tr} (\dot{W}_{u_n} \Lambda^{1/2} (\dot{W}_{u_n} \Lambda^{1/2})^\top). \] (14)

Note that \( V(0, 0, 0) = 0 \) and \( V(e, \dot{W}_\sigma, \dot{W}_{u_n}) > 0 \) for all \((e, \dot{W}_\sigma, \dot{W}_{u_n}) \neq (0, 0, 0)\). Now, differentiating (14) yields

\[ \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t)) = -e^T(t) R e(t) - 2e^T(t) P D \Lambda \dot{W}_\sigma^T(t) \sigma(x(t)) - 2e^T(t) P D \Lambda \dot{W}_{u_n}^T(t) u_n(t) \]

\[ + 2 \text{tr} \dot{W}_\sigma^T(t) \gamma_\sigma^{-1} \dot{W}_\sigma(t) \Lambda + 2 \text{tr} \dot{W}_{u_n}^T(t) \gamma_{u_n}^{-1} \dot{W}_{u_n}(t) \Lambda, \] (15)

where using (9) and (10) in (15) results in

\[ \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t)) = -e^T(t) R e(t) \leq 0. \] (16)

Since (16) holds, it follows from [Theorem 3.1, 14] that the solution \((e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t))\) of the closed-loop dynamical system is Lyapunov stable for all initial conditions and \( t \in \mathbb{R}_+ \). This implies that the terms \( e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t), \sigma(x(t)), \) and \( u_n(t) \) are bounded in (13), and hence, \( \dot{e}(t) \) is bounded for all \( t \in \mathbb{R}_+ \). Furthermore, since \( \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t)) = -2e^T(t) R \dot{e}(t) \), the boundedness of \( \dot{e}(t) \) results in the boundedness of \( \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t)) \). It now follows from Barbalat’s lemma [Lemma 4.1, 14] that

\[ \lim_{t \to \infty} \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t)) = 0, \] (17)
which consequently shows that \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \).

**Remark 2.** For the case when the nonlinear uncertain dynamical system given by (1) includes bounded exogenous disturbances, measurement noise, and/or the uncertainty in (1) cannot be perfectly parameterized, then (2) can be relaxed by considering

\[
\delta(t,x) = W(t)^T \sigma(x) + \varepsilon(t,x), \quad x \in \mathcal{D}_x,
\]

where \( W(t) \in \mathbb{R}^{s \times m} \) is an unknown time-varying weight matrix satisfying \( \|W(t)\|_F \leq w \) and \( \|\dot{W}(t)\|_F \leq \dot{w} \) with \( w \in \mathbb{R}_+ \) and \( \dot{w} \in \mathbb{R}_+ \) being unknown scalars, \( \sigma : \mathcal{D}_x \rightarrow \mathbb{R}^s \) is a sufficiently approximated basis function on \( x \in \mathcal{D}_x \) using universal approximation tools such as neural networks, \( \varepsilon : \mathbb{R}_+ \times \mathcal{D}_x \rightarrow \mathbb{R}^m \) is the system modeling error satisfying \( \|\varepsilon(t,x)\|_2 \leq \epsilon \) with \( \epsilon \in \mathbb{R}_+ \) being an unknown scalar, and \( \mathcal{D}_x \) is a compact subset of \( \mathbb{R}^n \). In this case, the weight update laws given by (9) and (10) can be replaced by

\[
\dot{\hat{W}}_\sigma(t) = \gamma_\sigma \text{Proj}[\hat{W}_\sigma(t), \sigma(x(t))e^T(t)PD], \quad \hat{W}_\sigma(0) = \hat{W}_{\sigma0}, \tag{19}
\]

\[
\dot{\hat{W}}_{un}(t) = \gamma_{un} \text{Proj}[\hat{W}_{un}(t), u_n(t)e^T(t)PD], \quad \hat{W}_{un}(0) = \hat{W}_{un0}, \tag{20}
\]

to guarantee the uniform boundedness of the system error state vector \( e(t) \) and the weight errors \( \hat{W}_\sigma(t) \) and \( \hat{W}_{un}(t) \), where Proj denotes the projection operator [16].

Even though Remark 1 highlights that \( x(t) \) asymptotically converges to \( x_r(t) \), \( x(t) \) can be *far different* from \( x_r(t) \) during the transient time (i.e., the learning phase of the adaptive controller). To address this problem, we introduce the artificial basis function approach in the next section for transient performance improvement.

### 3. Artificial Basis Functions for Transient Performance Improvement

In this section, we develop a new approach entitled *artificial basis functions* to improve the transient performance of the model reference adaptive control framework introduced in Section 2. In order to introduce our approach, we first write

\[
\dot{e}(t) = A_r e(t) + DA\left[u_a(t) + W_{\sigma}^T \sigma(x(t)) + W_{un}^T u_n(t)\right], \quad e(0) = e_0. \tag{21}
\]
using (4) and (7). Next, we add a new term “$W_a^T \sigma_a(t)$” to (21) as

$$\dot{e}(t) = A_r e(t) + D \Lambda \left[ u_a(t) + W_o^T \sigma(x(t)) + W_{u_n}^T u_n(t) + W_a^T \sigma_a(t) \right], \quad e(0) = e_0,$$

where we set $W_a \equiv 0$ in this term so that (21) and (22) are equivalent. Since the added term “$W_a^T \sigma_a(t)$” is zero by definition and it does not change the error dynamics, we call $W_a \in \mathbb{R}^{m \times q}$ as the artificial weighting and $\sigma_a(t) \in \mathbb{R}^q$ as the artificial basis function. Considering (22), we now let the new adaptive feedback control law be

$$u_a(t) = - \hat{W}_o^T(t) \sigma(x(t)) - \hat{W}_{u_n}^T(t) u_n(t) - \hat{W}_a^T(t) \sigma_a(t),$$

with $\hat{W}_a(t) \in \mathbb{R}^{m \times q}$, which yields

$$\dot{e}(t) = A_r e(t) - D \Lambda \left[ \hat{W}_o^T(t) \sigma(x(t)) + \hat{W}_{u_n}^T(t) u_n(t) + \hat{W}_a^T(t) \sigma_a(t) \right], \quad e(0) = e_0,$$

where $\hat{W}_o(t) \triangleq \hat{W}_o(t) - W_o \in \mathbb{R}^{s \times m}$, $\hat{W}_{u_n}(t) \triangleq \hat{W}_{u_n}(t) - W_{u_n} \in \mathbb{R}^{m \times m}$, and $\hat{W}_a(t) \triangleq \hat{W}_a(t) - W_a \in \mathbb{R}^{m \times q}$ (note that $\hat{W}_a(t) = \hat{W}_a(t)$ since $W_a \equiv 0$). In the rest of this section, we choose the update laws for the artificial basis function $\sigma_a(t)$ and the artificial weight update law $\hat{W}_a(t)$ in the proposed adaptive feedback control law (23) in order to improve the transient performance without sacrificing the asymptotic stability of the closed-loop system error dynamics in (24). To this end, the following two theorems present the main results of this section.

**Theorem 1.** Consider the system error dynamics given by (24) and the artificial basis function update law given by

$$\dot{\sigma}_a(t) = k \hat{W}_a(t) (D^T D)^{-1} D^T [\dot{e}(t) - A_r e(t)], \quad \sigma_a(0) = \sigma_{a0} \neq 0.$$

where $k \in \mathbb{R}_+$. Then, (25) is constructed through the negative gradient of

$$J(\cdot) = \frac{1}{2} \| \Lambda^{1/2} (\hat{W}_o^T(t) \sigma(x(t)) + \hat{W}_{u_n}^T(t) u_n(t) + \hat{W}_a^T(t) \sigma_a(t)) \|_2^2,$$

with respect to $\sigma_a(t)$.

**Proof.** Consider the cost function given by (26) and note that its gradient with respect to $\sigma_a(t)$ has the form
\[
\frac{\partial [-J(\cdot)]}{\partial \sigma_a(t)} = -\tilde{W}_a(t) \Lambda \left[ \tilde{W}_a^T(t) \sigma(x(t)) + \tilde{W}_{u_n}^T u_n(t) + \tilde{W}_a^T(t) \sigma_a(t) \right] \\
= -\tilde{W}_a(t) \Lambda \left[ \tilde{W}_a^T(t) \sigma(x(t)) + \tilde{W}_{u_n}^T u_n(t) + \tilde{W}_a^T(t) \sigma_a(t) \right],
\]

(27)
since \( W_a \equiv 0 \). Using the idea presented in [17–19], we now construct the update law for the artificial basis function as
\[
\dot{\sigma}_a(t) = k \frac{\partial [-J(\cdot)]}{\partial \sigma_a(t)} \\
= -k \tilde{W}_a(t) \Lambda \left[ \tilde{W}_a^T(t) \sigma(x(t)) + \tilde{W}_{u_n}^T u_n(t) + \tilde{W}_a^T(t) \sigma_a(t) \right], \quad \sigma_a(0) = \sigma_{a0}.
\]
(28)

Here, one can notice that (28) has unknown terms (i.e., \( \Lambda \) is unknown and the first two terms inside the brackets are unknown since \( W_a, W_{u_n} \in \mathbb{R}^{s \times m} \) are unknown in \( \tilde{W}_a(t) = \tilde{W}_a(t) - W_a \in \mathbb{R}^{s \times m} \) and \( \tilde{W}_{u_n}(t) = \tilde{W}_{u_n}(t) - W_{u_n} \in \mathbb{R}^{m \times m} \), respectively), and hence, it can not be implemented. To address this problem, (24) can be rewritten as
\[
-\Lambda \left[ \tilde{W}_a^T(t) \sigma(x(t)) + \tilde{W}_{u_n}^T u_n(t) + \tilde{W}_a^T(t) \sigma_a(t) \right] = (D^T D)^{-1} D^T \left[ \dot{e}(t) - A_r e(t) \right].
\]
(29)
Since (28) along with (29) leads to the artificial basis function update law given by (25), it follows that (25) is the negative gradient of (26).

\[\blacksquare\]

Remark 3. The unknown magnitude of the mismatch term
\[
\tilde{W}_a^T(t) \sigma(x(t)) + \tilde{W}_{u_n}^T u_n(t) + \tilde{W}_a^T(t) \sigma_a(t),
\]
in (24) can lead to a large deviation of the state from the reference state during the learning phase of the adaptive controller given by (23). From this standpoint, the proposed artificial basis function allows to shape the system error by suppressing the mismatch term (30) in (24) due to gradient optimization, since it is constructed to be the negative gradient of (26) with respect to \( \sigma_a(t) \) (see, for example, [17–19] and references included therein on other applications of gradient optimization in the context of adaptive control). Therefore, by adjusting \( k \) in (25), the uncertain dynamical system response and the reference system response can be made close to each other for all time including the transient phase.

Remark 4. Even though the artificial basis function update law given by (25) has the time derivative of the system error on its right hand side, we will see in Corollary 1 of the next section
that we can use an equivalent form of this update law *without requiring* this time derivative for real-world applications.

In Theorem 1, we developed an update law for the artificial basis function in order to improve the transient performance of the system error dynamics. In the next theorem, we choose an appropriate artificial weight update law \( \hat{W}_a(t) \) (and also update laws for \( \hat{W}_\sigma(t) \) and \( \hat{W}_{u_n}(t) \)) to guarantee the asymptotic stability of the closed-loop system error dynamics in (24). We will also show the transient performance bounds satisfied by the system error dynamics.

**Theorem 2.** Consider the nonlinear uncertain dynamical system given by (1), the reference system given by (4), the feedback control law given by (23) along with the update laws given by (25) and

\[
\begin{align*}
\dot{\hat{W}}_\sigma(t) &= \gamma_\sigma \left( \sigma(x(t)) e^T(t) PD + \mu_\sigma(x(t)) \sigma_a^T(t) \hat{W}_a(t) \right), \quad \hat{W}_\sigma(0) = \hat{W}_{\sigma 0}, \quad (31) \\
\dot{\hat{W}}_{u_n}(t) &= \gamma_{u_n} \left[ u_n(t) e^T(t) PD + \mu u_n(t) \sigma_a^T(t) \hat{W}_a(t) \right], \quad \hat{W}_{u_n}(0) = \hat{W}_{u_n 0}, \quad (32) \\
\dot{\hat{W}}_a(t) &= \gamma_a \sigma_a(t) e^T(t) PD, \quad \hat{W}_a(0) = \hat{W}_{a 0} \neq 0, \quad (33)
\end{align*}
\]

where \( \gamma_\sigma \in \mathbb{R}_+, \gamma_{u_n} \in \mathbb{R}_+, \gamma_a \in \mathbb{R}_+, \) and \( \mu \in \mathbb{R}_+ \). Then, the solution \( (e(t), \hat{W}_\sigma(t), \hat{W}_{u_n}(t), \hat{W}_a(t), \sigma_a(t)) \) of the closed-loop dynamical system is Lyapunov stable for all initial conditions and \( t \in \mathbb{R}_+ \), and \( \lim_{t \to \infty} e(t) = 0 \). In addition, the system error dynamics satisfy the transient performance bounds given by

\[
\| e(t) \|_{L_\infty} \leq \sqrt{\epsilon/\lambda_{\text{min}}(P)}, \quad (34)
\]

where

\[
\epsilon \triangleq \lambda_{\text{max}}(P) \| e(0) \|^2_2 + \gamma_\sigma^{-1} \| \hat{W}_{\sigma 0} \Lambda^{1/2} \|^2_F + \gamma_{u_n}^{-1} \| \hat{W}_{u_n 0} \Lambda^{1/2} \|^2_F + \gamma_a^{-1} \| \hat{W}_{a 0} \Lambda^{1/2} \|^2_F + k^{-1} \| \sigma_a(0) \|^2_2. \quad (35)
\]

**Proof.** Consider the Lyapunov function candidate given by

\[
\mathcal{V}(e, \hat{W}_\sigma, \hat{W}_{u_n}, \hat{W}_a, \sigma_a) = e^T P e + \gamma_\sigma^{-1} \text{tr} \left( \hat{W}_\sigma \Lambda^{1/2} \right)^T (\hat{W}_\sigma \Lambda^{1/2}) + \gamma_{u_n}^{-1} \text{tr} \left( \hat{W}_{u_n} \Lambda^{1/2} \right)^T (\hat{W}_{u_n} \Lambda^{1/2}) + \gamma_a^{-1} \text{tr} \left( \hat{W}_a \Lambda^{1/2} \right)^T (\hat{W}_a \Lambda^{1/2}) + \mu k^{-1} \sigma_a^T \sigma_a, \quad (36)
\]

where \( \mathcal{V}(0,0,0,0,0) = 0 \) and \( \mathcal{V}(e, \hat{W}_\sigma, \hat{W}_{u_n}, \hat{W}_a, \sigma_a) > 0 \) for all \( (e, \hat{W}_\sigma, \hat{W}_{u_n}, \hat{W}_a, \sigma_a) \neq (0,0,0,0,0) \). It follows that
\[ \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t)) \]
\[ = 2e^T(t)P\dot{e}(t) + 2\gamma_\sigma^{-1}\text{tr} \dot{W}_\sigma^T(t)\Lambda \dot{W}_\sigma(t) + 2\gamma_{u_0}^{-1}\text{tr} \dot{W}_{u_0}(t)\Lambda \dot{W}_{u_0}(t) \]
\[ + 2\gamma_a^{-1}\text{tr} \dot{W}_a^T(t)\Lambda \dot{W}_a(t) + 2\mu k^{-1}\sigma_a(t)\dot{\sigma}_a(t) \]
\[ = -e^T(t)Re(t) - 2\mu \sigma_a^T(t)\dot{W}_a(t)\Lambda \dot{W}_a(t)\sigma_a(t) \]
\[ \leq -e^T(t)Re(t) \leq 0, \quad t \in \mathbb{R}_+, \] (37)

which guarantees that \((e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t))\) is Lyapunov stable, and hence, is bounded for all \(t \in \mathbb{R}_+\). Since \(\sigma(x(t)), u_0(t), \) and \(\sigma_a(t)\) are bounded for all \(t \in \mathbb{R}_+\), it follows from (24) that \(\dot{e}(t)\) is bounded, and hence, \(\dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t))\) is bounded for all \(t \in \mathbb{R}_+\). Then according to Barbalat’s lemma

\[ \lim_{t \to \infty} \dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t)) = 0, \] (38)

which consequently shows that \(e(t) \to 0\) as \(t \to \infty\).

Additionally, because \(\dot{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t)) \leq 0\) for \(t \in \mathbb{R}_+\), this implies that

\[ \mathcal{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t)) \leq \mathcal{V}(e(0), \dot{W}_{\sigma_0}, \dot{W}_{u_{00}}, \dot{W}_{a0}, \sigma_a(0)). \] (39)

Then using the inequalities

\[ \lambda_{\min}(P)\|e(t)\|_2^2 \leq \mathcal{V}(e(t), \dot{W}_\sigma(t), \dot{W}_{u_0}(t), \dot{W}_a(t), \sigma_a(t)) \] (40)

and

\[ \mathcal{V}(e(0), \dot{W}_{\sigma_0}, \dot{W}_{u_{00}}, \dot{W}_{a0}, \sigma_a(0)) \leq \lambda_{\max}(P)\|e(0)\|_2^2 + \gamma_\sigma^{-1}\|\dot{W}_{\sigma_0}\|_F^2 + \gamma_{u_0}^{-1}\|\dot{W}_{u_{00}}\|_F^2 \]
\[ + \gamma_a^{-1}\|\dot{W}_{a0}\|_F^2 + k^{-1}\|\sigma_a(0)\|_2^2 \] (41)

in (39) results in

\[ \|e(t)\|_2 \leq \sqrt{\epsilon/\lambda_{\min}(P)}. \] (42)

Since \(\|\cdot\|_\infty \leq \|\cdot\|_2\), and this bound is uniform, then (42) yields

\[ \|e_\tau(t)\|_\infty \leq \sqrt{\epsilon/\lambda_{\min}(P)} \] (43)

therefore, (34) is a direct consequence of (43) because (43) holds uniformly in \(\tau\). \[\blacksquare\]
4. Practical Considerations

In Theorem 1 of the previous section, it is noted that (25) presents the update law for the artificial basis function that contains the time derivative of the system error in its right hand side. In practice, it is desired to remove this term from the update law. Motivating from the methods used in [20], the next corollary presents an equivalent form of the update law in (25), but without the time derivative of the system error.

**Corollary 1.** The update law for the artificial basis function given by (25) is identical to

$$
\sigma_a(t) = \sigma_a(0) + k \left[ \hat{W}_a(t)(D^T D)^{-1} D^T e(t) - \hat{W}_a(0)(D^T D)^{-1} D^T e(0) \right]
$$

(44)

**Proof.** To show that (25) is equivalent to (44), we first integrate both sides of (25) as

$$
\int_0^t \frac{d\sigma_a}{dr} dr = k \left[ \int_0^t \hat{W}_a(\tau)(D^T D)^{-1} D^T \frac{de(\tau)}{d\tau} d\tau - \int_0^t \hat{W}_a(\tau)(D^T D)^{-1} D^T A e(\tau) d\tau \right],
$$

(45)

where $k$ is constant. The first term on the right hand side of (45) can then be manipulated using integration by parts of the form

$$
\int U dV = UV - \int dUV,
$$

(46)

with $U = \hat{W}_a(\tau)(D^T D)^{-1} D^T$ and $V = e(\tau)$, and respectively $dU = \hat{W}_a(\tau)(D^T D)^{-1} D^T d\tau$ and $dV = \frac{de(\tau)}{d\tau} d\tau$. This produces an equivalent term of the form

$$
\hat{W}_a(t)(D^T D)^{-1} D^T e(t) - \hat{W}_a(0)(D^T D)^{-1} D^T e(0) - \int_0^t \hat{W}_a(\tau)(D^T D)^{-1} D^T e(\tau) d\tau,
$$

(47)

that does not contain the time derivative of the system error. We can further expand this using (33) as

$$
\hat{W}_a(t)(D^T D)^{-1} D^T e(t) - \hat{W}_a(0)(D^T D)^{-1} D^T e(0) - \int_0^t \gamma_a \sigma_a(\tau) e^T(\tau) P D(D^T D)^{-1} D^T e(\tau) d\tau.
$$

(48)

Using (48) instead of the first term on the right hand side of (38) and integrating the left hand
\[
\sigma_a(t) - \sigma_a(0) = k \left[ \tilde{W}_a(t)(D^T D)^{-1} D^T e(t) - \tilde{W}_a(0)(D^T D)^{-1} D^T e(0) \right]
- \int_0^t \gamma_a \sigma_a(\tau) e^T(\tau) P D (D^T D)^{-1} D^T e(\tau) d\tau
- \int_0^t \tilde{W}_a(\tau)(D^T D)^{-1} D^T e(\tau) d\tau
\]
(49)

Adding the initial condition \(\sigma_a(0)\) to both sides concludes the proof.

Since the update law of the artificial basis function given by (25), or equivalently (44), is derived through a gradient optimization procedure, it may (or may not) induce oscillations to the system response as the value of \(k\) gets large. Even though we did not observe such an oscillative system response in the illustrative example of the next section (as well as in applications to various uncertain dynamical systems); if such a situation happens, then it is of practical importance to robustify the proposed approach against such oscillative (i.e., high-frequency) dynamical system content. To this end, one can adopt, for example, the low-frequency learning idea of [1] to achieve both improved transient performance and smooth system behavior. This is highlighted in the next corollary.

**Corollary 2.** Consider the nonlinear uncertain dynamical system given by (1), the reference system given by (4), the feedback control law given by (23) along with the weight update laws given by (31), (32), (33),
\[
\dot{\sigma}_a(t) = k \tilde{W}_a(t)(D^T D)^{-1} D^T \left[ \dot{e}(t) - A_r e(t) \right] - c_1 (\sigma_a(t) - \sigma_{af}(t)),
\]
(50)
and
\[
\dot{\sigma}_{af}(t) = -c_2 (\sigma_{af}(t) - \sigma_a(t)),
\]
(51)
where \(c_1 \in \mathbb{R}_+\), and \(c_2 \in \mathbb{R}_+\). Then the solution \((e(t), \tilde{W}_\sigma(t), \tilde{W}_{un}(t), \tilde{W}_a(t), \sigma_a(t), \sigma_{af}(t))\) of the closed-loop system given by (24), (31), (32), (33), (50), and (51) is Lyapunov stable for all initial conditions and \(t \in \mathbb{R}_+\), and \(\lim_{t \to \infty} e(t) = 0\).

**Proof.** Considering the Lyapunov function candidate given by
\[
\mathcal{V}(e, \tilde{W}_\sigma, \tilde{W}_{un}, \tilde{W}_a, \sigma_a, \sigma_{af}) = e^T P e + \gamma_{\sigma}^{-1} \text{tr} \left( \tilde{W}_\sigma \Lambda^{1/2} \right)^T (\tilde{W}_\sigma \Lambda^{1/2}) + \gamma_{un}^{-1} \text{tr} \left( \tilde{W}_{un} \Lambda^{1/2} \right)^T (\tilde{W}_{un} \Lambda^{1/2})
+ \gamma_a^{-1} \text{tr} \left( \tilde{W}_a \Lambda^{1/2} \right)^T (\tilde{W}_a \Lambda^{1/2}) + \mu k^{-1} \sigma_a^T \sigma_a + c_2^{-1} \mu k^{-1} c_1 \sigma_{af}^T \sigma_{af},
\]
(52)
where \( \mathcal{V}(0,0,0,0,0) = 0 \) and \( \mathcal{V}(e, \tilde{W}_a, \tilde{W}_{un}, \tilde{W}_a, \sigma_a, \sigma_{af}) > 0 \) for all \( (e, \tilde{W}_a, \tilde{W}_{un}, \tilde{W}_a, \sigma_a, \sigma_{af}) \neq (0,0,0,0,0,0) \). Differentiating (52) along the closed-loop system trajectories of (24), (31), (32), (33), (50), and (51) yields

\[
\dot{\mathcal{V}}(e(t), \tilde{W}_a(t), \tilde{W}_{un}(t), \tilde{W}_a(t), \sigma_a(t), \sigma_{af}(t)) = -e^T(t)Re(t) - 2\mu \sigma_a^T(t) \tilde{W}_a(t) \Lambda \tilde{W}_a^T(t) \sigma_a(t) - 2\mu k^{-1} c_1 (\sigma_a(t) - \sigma_{af}(t))^T (\sigma_a(t) - \sigma_{af}(t))
\leq -e^T(t)Re(t) \leq 0, \quad t \in \overline{\mathbb{R}}_+.
\tag{53}
\]

Hence it is guaranteed that the closed-loop dynamical system given by (24), (31), (32), (33), (50), and (51) is Lyapunov stable, and therefore bounded for all \( t \in \overline{\mathbb{R}}_+ \). Since \( \sigma(x(t)), u_n(t), \) and \( \sigma_a(t) \) are bounded for all \( t \in \overline{\mathbb{R}}_+ \), it follows from (24) that \( \dot{e}(t) \) is bounded, and hence, \( \dot{\mathcal{V}}(e(t), \tilde{W}_a(t), \tilde{W}_{un}(t), \tilde{W}_a(t), \sigma_a(t), \sigma_{af}(t)) \) is bounded for all \( t \in \overline{\mathbb{R}}_+ \). It then follows from Barbalat's lemma that

\[
\lim_{t \to \infty} \mathcal{V}(e(t), \tilde{W}_a(t), \tilde{W}_{un}(t), \tilde{W}_a(t), \sigma_a(t), \sigma_{af}(t)) = 0,
\tag{54}
\]

which consequently shows that \( e(t) \to 0 \) as \( t \to \infty \). \( \blacksquare \)

It should be noted that using similar steps highlighted in Corollary 1, (50) can be equivalently written as

\[
\sigma_a(t) = \sigma_a(0) + k \left\{ \tilde{W}_a(t)(D^T D)^{-1}D^T e(t) - \tilde{W}_a(0)(D^T D)^{-1}D^T e(0) \right\}
- \int_0^t \gamma_a \sigma_a(\tau) e^T(\tau) P D (D^T D)^{-1} D^T e(\tau) d\tau - \int_0^t \tilde{W}_a(\tau)(D^T D)^{-1} D^T A_x e(\tau) d\tau
- \int_0^t c_1 (\sigma_a(\tau) - \sigma_{af}(\tau)) d\tau,
\tag{55}
\]

without the time derivative of the system error.

**Remark 5.** Following the discussion stated before Corollary 2, the added term to the right hand side of (50) (or equivalently (55)) filters out possible high-frequency dynamical system content in \( \sigma_a(t) \) (while preserving asymptotic stability of the system error dynamics) as one increases the design parameter \( c_1 \) for driving the trajectories of \( \sigma_a(t) \) closer to the trajectories of \( \sigma_{af}(t) \). Note that the frequency content of such possible high-frequency oscillations that
one desires to suppress is defined through the design parameter $c_2$ in (51), which denotes the bandwidth of $\sigma_{af}(t)$ (we refer to [1] for additional technical details and discussions).

5. Illustrative Example

In order to illustrate the proposed adaptive control architecture based on artificial basis functions, consider the nonlinear dynamical system representing a controlled wing rock dynamics model given by

$$\dot{x}_1(t) = x_2(t), \quad x_1(0) = 0, \quad (56)$$

$$\dot{x}_2(t) = \Lambda u(t) + \delta(x(t)), \quad x_2(0) = 0, \quad (57)$$

where $x_1$ represents the roll angle in radians and $x_2$ represents the roll rate in radians per second. In (57), $\delta(x)$ represents an uncertainty of the form $\delta(x) = a_1x_1 + a_2x_2 + a_3|x_1|x_2 + a_4|x_2|x_2 + a_5|x_1^3$, where $a_i$, $i = 1, \ldots, 5$, are unknown parameters that are derived from the aircraft aerodynamic coefficients. For our numerical example, we set $a_1 = 0.1414$, $a_2 = 0.5504$, $a_3 = -0.0624$, $a_4 = 0.0095$, $a_5 = 0.0215$, and $\Lambda = 0.5$. We choose $K_1 = [-0.16, -0.57]$ and $K_2 = 0.16$ for the nominal controller design that yields to a reference system with a natural frequency of $\omega_n = 0.40$ rad/s and a damping ratio $\zeta = 0.707$. For the standard adaptive controller design given by (5), (6), (8), (9), and (10), $\sigma(x) = [x_1, x_2, |x_1|x_2, |x_2|x_2, x_1^3]^T$ is used for the basis function and we set $R = I_2$. For the proposed adaptive controller design given by (5), (6), (23), (31), (32), (33), and (44), we use the same basis function and $R$ as well as $q = 1$ is chosen implying the artificial basis function is one-dimensional.

Figures 1–6 compare the standard control design with the proposed design for a given square-wave tracking command. In particular, Figures 1–3 show the standard model reference adaptive control design with adaptation gains of $\gamma_{\sigma} = \gamma_{u_n} = 0.5$, 10, and 50, respectively. The higher adaptation gain used in Figure 3 yields to a better system performance pertaining to the roll angle, but it is not acceptable due to the oscillative content in the roll rate response and the control response. Figures 4–6 show the proposed design with the smallest adaptation gain used for the standard design, i.e., $\gamma_{\sigma} = 0.5$. As we increase $k$ from 5 to 25, and then 25 to 100, these figures
clearly highlight the improvement on the transient performance due to the nature of gradient optimization. In other words, the results with the proposed adaptive controller design, especially the ones in Figures 5 and 6, are superior as compared with the standard ones.

6. Conclusion

To contribute to the previous studies in adaptive control theory, we investigated a new approach based on artificial basis functions. Specifically, we showed that these functions, which are constructed based on gradient optimization, can improve the transient response of an adaptively controlled system, and hence, can be used to achieve predictable closed-loop system performance. We further discussed in detail regarding the practical aspects of the proposed design and included a detailed illustrative example. Future research will include extensions to uncertain dynamical systems with limited state information (i.e., output feedback adaptive control), state constraints, and control constraints.

Figure 1: Standard adaptive control performance with (5), (6), (8), (9), and (10) for a given square-wave tracking command ($\gamma_\sigma = 0.5$ and $\gamma_{\mu_n} = 0.5$).
\textbf{Figure 2}: Standard adaptive control performance with (5), (6), (8), (9), and (10) for a given square-wave tracking command ($\gamma_{\sigma} = 10$ and $\gamma_{u_{\kappa}} = 10$).

\textbf{Figure 3}: Standard adaptive control performance with (5), (6), (8), (9), and (10) for a given square-wave tracking command ($\gamma_{\sigma} = 50$ and $\gamma_{u_{\kappa}} = 50$).
Figure 4: Proposed adaptive control performance with (5), (6), (23), (31), (32), (33), and (44) for a given square-wave tracking command ($\hat{W}_{a0} = 0.1, \sigma_{a0} = 0.1, \gamma_{\sigma} = 0.5, \gamma_{u_n} = 1, \gamma_a = 1, k = 5$, and $\mu = 1$).

Figure 5: Proposed adaptive control performance with (5), (6), (23), (31), (32), (33), and (44) for a given square-wave tracking command ($\hat{W}_{a0} = 0.1, \sigma_{a0} = 0.1, \gamma_{\sigma} = 0.5, \gamma_{u_n} = 1, \gamma_a = 1, k = 25$, and $\mu = 1$).
Figure 6: Proposed adaptive control performance with (5), (6), (23), (31), (32), (33), and (44) for a given square-wave tracking command ($\hat{W}_{\alpha_0} = 0.1, \sigma_{\alpha_0} = 0.1, \gamma_{\sigma} = 0.5, \gamma_{u_n} = 1, \gamma_{a} = 1, k = 100, \text{and } \mu = 1$).
References


