Performance Oriented Adaptive Architectures with Guaranteed Bounds

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While adaptive control has been used in numerous applications to achieve given system stabilization or command following criteria, the ability to obtain a predictable transient performance is a challenging problem when there is no a priori knowledge about system uncertainties (e.g., their upper bounds and/or domains). In order to address this problem, a new method is presented in [1, 2] utilizing artificial basis functions in the update law of an adaptive control design. This approach is predicated on a gradient minimization procedure and achieves a predictable transient performance without inducing oscillations in the system response as the constant gain due to the nature of this minimization approach is judiciously increased. However, selection of this gain is problem dependent and may need to be adjusted for each different design.

To address this problem, we present a new approach which has an ability to auto-tune an adaptive control design with artificial basis functions employed when the controlled system is about to violate a given design constraint on error dynamics (i.e., only when it is necessary). In particular, our approach is based on a controller architecture that allows the assignment of a priori known (user-defined) transient performance bounds. These bounds are constructed through a restricted potential function approach [3] that yields to an error dependent gain to adjust system performance for time instants when it is required to meet given design criteria. In addition to the theoretical results based on Lyapunov stability arguments highlighting transient performance of an uncertain system that stays within a priori given performance bounds, an illustrative example is provided to demonstrate the efficacy of the proposed framework.

I. Introduction

Even though adaptive control has been applied to obtain stabilization and command tracking requirements without relying heavily on accurate mathematical system models, the ability to guarantee transient performance remains a critical issue [4, 5]. One method to address this problem is to use a high learning rate in the update law which minimizes the worst-case system error between an uncertain dynamical system and a given reference model. However, an update law subject to a high gain all the time may not be necessary. This is because an adaptive controller can yield to a desired system response with an acceptable transient performance when there is no significant uncertainty in the system. Furthermore, even in the presence of uncertainties, which might be due to reconfiguration, deployment of a payload, docking, or structural damage, an adaptive controller is required to resort to a high-gain only when it is absolutely necessary (e.g., when it is critical not to exceed a given angle of attack requirement) in order not to utilize available system bandwidth.
continuously. Therefore, the gain of an update law can be adjusted in response to a system performance only when necessary to yield guaranteed transient response properties without using high-gain all the time.

To address predictable transient performance of adaptive controllers, a new method is presented in [1,2] utilizing artificial basis functions in the update law of an adaptive control design. This approach is predicated on a gradient minimization procedure and achieves a predictable transient performance without inducing oscillations in the system response as the constant gain due to the nature of this minimization approach is judiciously increased. However, selection of this gain is problem dependent and may need to be adjusted for each different design.

To address this problem, we present a new approach which has an ability to auto-tune an adaptive control design with artificial basis functions employed, when the controlled system is about to violate a given design constraint on error dynamics (i.e., only when it is necessary). In particular, our approach is based on a controller architecture that allows the assignment of a priori known (user-defined) transient performance bounds. These bounds are constructed through a restricted potential function approach [3] that yields to an error dependent gain to adjust system performance for time instants when it is required to meet given design criteria. In addition to the theoretical results based on Lyapunov stability arguments highlighting transient performance of an uncertain system that stays within a priori given performance bounds, an illustrative example is provided to demonstrate the efficacy of the proposed framework.

The notation used in this paper is fairly standard. Specifically, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_+^n \)) denotes the set of positive (resp. non-negative-definite) real numbers, \( \mathbb{R}_+^{n \times n} \) (resp. \( \mathbb{R}_+^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (resp. non-negative-definite) real matrices, \( \mathbb{S}^{n \times n} \) denotes the set of \( n \times n \) symmetric real matrices, \( \mathbb{D}^{n \times n} \) denotes the set of \( n \times n \) real matrices with diagonal scalar entries, \( (\cdot)^T \) denotes transpose, \( (\cdot)^{-1} \) denotes inverse, \( \text{tr}(\cdot) \) denotes the trace operator, and \( \triangleq \) denotes equality by definition.

## II. Preliminaries

We now briefly state the standard model reference control problem. Specifically, consider the nonlinear uncertain dynamical system given by

\[
\dot{x}(t) = Ax(t) + Bu(t) + D\delta(x(t)), \quad x(0) = x_0,
\]

where \( x(t) \in \mathbb{R}^n \) is the state vector available for feedback, \( u(t) \in \mathbb{R}^m \) is the control input restricted to the class of admissible controls consisting of measurable functions, \( \delta : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is an uncertainty, \( A \in \mathbb{R}^{n \times n} \) is a known system matrix, \( B \in \mathbb{R}^{n \times m} \) is an unknown control input matrix, \( D \in \mathbb{R}^{n \times m} \) is a known uncertainty input matrix, and the pair \( (A, B) \) is controllable.

**Assumption 1.** The uncertainty in (1) is parameterized as

\[
\delta(x) = W^T \sigma(x), \quad x \in \mathbb{R}^n,
\]

where \( W \in \mathbb{R}^{s \times m} \) is an unknown weight matrix and \( \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^s \) is a known basis function of the form \( \sigma(x) = [\sigma_1(x), \sigma_2(x), \ldots, \sigma_s(x)]^T \). In addition, the unknown control input matrix is parameterized as

\[
B = DA,
\]

where \( \det(D^TD) \neq 0 \) and \( A \in \mathbb{R}_+^{m \times m} \cap D^{m \times m} \) is an unknown control effectiveness matrix.

Next, consider the reference system given by

\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0},
\]

where \( x_r(t) \in \mathbb{R}^n \) is the reference state vector, \( c(t) \in \mathbb{R}^m \) is a given uniformly continuous bounded command \( (c(t) \equiv 0 \text{ for stabilization}) \), \( A_r \in \mathbb{R}_+^{n \times n} \) is the Hurwitz reference system matrix, and \( B_r \in \mathbb{R}_+^{n \times m} \) is the command input matrix. The objective of the model reference adaptive control problem is to construct a feedback control law \( u(t) \) such that the state vector \( x(t) \) asymptotically follows the reference state vector \( x_r(t) \) subject to Assumption 1.

For the purpose of stating the preliminaries associated with the model reference adaptive control problem,
Consider the feedback control law given by

$$u(t) = u_n(t) + u_a(t),$$

(5)

where $u_n(t) \in \mathbb{R}^m$ is the nominal feedback control law and $u_a(t) \in \mathbb{R}^m$ is the adaptive feedback control law. Additionally, let the nominal feedback control law be given by

$$u_a(t) = K_1 x(t) + K_2 e(t),$$

(6)

where $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ are the nominal feedback and the nominal feedforward gains, respectively, such that $A_t = A + DK_1$, $B_t = DK_2$, and det($K_2$) $\neq 0$ holds. Now, using (5) and (6) in (1) along with Assumption 1 yields

$$\dot{x}(t) = A_t x(t) + B_t c(t) + D A [u_n(t) + W_\sigma^T \sigma(x(t)) + W_{u_n}^T u_n(t)],$$

(7)

where $W_\sigma \triangleq W \Lambda^{-1} \in \mathbb{R}^{s \times m}$ and $W_{u_n} \triangleq |I - \Lambda^{-1}| \in \mathbb{R}^{m \times m}$.

Next, let the adaptive feedback control law be given by

$$u_a(t) = -\dot{W}_\sigma^T(t) \sigma(x(t)) - \dot{W}_{u_n}^T u_n(t),$$

(8)

where $\dot{W}_\sigma(t) \in \mathbb{R}^{s \times m}$ and $\dot{W}_{u_n}(t) \in \mathbb{R}^{m \times m}$ are the estimates of $W_\sigma$ and $W_{u_n}$, respectively, satisfying the weight update laws

$$\dot{W}_\sigma(t) = \gamma_\sigma \sigma(x(t)) e^T(t) P D, \quad \dot{W}_{u_n}(t) = \gamma_{u_n} u_n(t) e^T(t) P D,$$

(9, 10)

where $\gamma_\sigma \in \mathbb{R}^{s \times s}$ and $\gamma_{u_n} \in \mathbb{R}^{m \times m}$ are the learning rate matrices, $e(t) \triangleq x(t) - x_t(t)$ is the system error state vector, and $P \in \mathbb{R}^{n \times n}$ is a solution of the Lyapunov equation

$$0 = A_t^T P + PA_t + R,$$

(11)

where $R \in \mathbb{R}^{n \times n}$ can be viewed as an additional learning rate. Note that because $A_t$ is Hurwitz, it follows from the converse Lyapunov theory [6] that there exists a unique $P$ satisfying (11) for a given $R$.

Now, using (8) in (7) yields

$$\dot{x}(t) = A_t x(t) + B_t c(t) - D A [\dot{W}_\sigma^T(t) \sigma(x(t)) + \dot{W}_{u_n}^T(t) u_n(t)],$$

(12)

and the system error dynamics is given using (4) and (12) as

$$\dot{e}(t) = A_t c(t) - D A [\dot{W}_\sigma^T(t) \sigma(x(t)) + \dot{W}_{u_n}^T(t) u_n(t)], \quad e(0) = e_0,$$

(13)

where $\dot{W}_\sigma(t) \triangleq \dot{W}_\sigma(t) - W_\sigma \in \mathbb{R}^{s \times m}$ and $\dot{W}_{u_n}(t) \triangleq \dot{W}_{u_n}(t) - W_{u_n} \in \mathbb{R}^{m \times m}$. Note that the weight update laws given by (9) and (10) can be derived using Lyapunov analysis by considering the Lyapunov function candidate (see, for example, [7])

$$V(e, \dot{W}_\sigma, \dot{W}_{u_n}) = e^T P e + \gamma^{-1}_\sigma \text{tr} (\dot{W}_\sigma \Lambda^{1/2})^T (\dot{W}_\sigma \Lambda^{1/2}) + \gamma_{u_n}^{-1} \text{tr} (\dot{W}_{u_n} \Lambda^{1/2})^T (\dot{W}_{u_n} \Lambda^{1/2}),$$

(14)

to show the system error state vector $e(t)$ and the weight errors $\dot{W}_\sigma(t)$ and $\dot{W}_{u_n}(t)$ are Lyapunov stable, and are therefore bounded and $e(t) \to 0$ as $t \to \infty$.

### III. Overview of Artificial Basis Function Approach for Adaptive Control

In this section we overview the artificial basis function control architecture of [1, 2]. From the model reference adaptive control framework introduced in the previous section, we can develop the approach based on artificial basis functions to shape the transient system performance. Specifically, we first use (4), (7), and
\[ e(t) = x(t) - x_r(t) \] to write
\[ \dot{e}(t) = A_r e(t) + DA\begin{bmatrix} u_a(t) + W^T_\sigma \sigma(x(t)) + W^T_{u_n} u_n(t) \end{bmatrix}, \quad e(0) = e_0. \tag{15} \]

Note that (15) can be equivalently written as
\[ \dot{e}(t) = A_r e(t) + DA\begin{bmatrix} u_a(t) + W^T_\sigma \sigma(x(t)) + W^T_{u_n} u_n(t) + W^T_{a} \sigma_a(t) \end{bmatrix}, \quad e(0) = e_0, \tag{16} \]

by noting \( W_a \equiv 0 \), where we call \( W_a \in \mathbb{R}^{m \times q} \) and \( \sigma_a(t) \in \mathbb{R}^q \) the artificial weighting and artificial basis function, respectively, then \( W^T_a \sigma_a(t) \equiv 0 \). Considering (16), let the new adaptive feedback control law be
\[ u_a(t) = -\dot{W}_a^T(t) \sigma(x(t)) - W^T_{u_n} u_n(t) - \dot{W}_a^T(t) \sigma_a(t), \tag{17} \]
with \( \dot{W}_a(t) \in \mathbb{R}^{m \times q} \), which yields
\[ \dot{e}(t) = A_r e(t) - DA\begin{bmatrix} \dot{W}_a^T(t) \sigma(x(t)) + \dot{W}_{u_n}^T u_n(t) + \dot{W}_a^T(t) \sigma_a(t) \end{bmatrix}, \quad e(0) = e_0, \tag{18} \]
where \( \dot{W}_a(t) \equiv \dot{W}_a(t) - W_a \in \mathbb{R}^{n \times m}, \dot{W}_{u_n}(t) \equiv \dot{W}_{u_n}(t) - W_{u_n} \in \mathbb{R}^{m \times n}, \) and \( \dot{W}_a(t) \equiv \dot{W}_a(t) - W_a \in \mathbb{R}^{m \times q} \) (note that \( \dot{W}_a(t) = \dot{W}_a(t) \) since \( W_a \equiv 0 \)). Using the above redefined system error dynamics, the rest of this section presents the main results from [1] (also see this reference for the proofs of stated theorems).

**Theorem 1.** Consider the system error dynamics given by (18) and the artificial basis function update law given by
\[ \dot{\sigma}_a(t) = k \dot{W}_a(t)(D^T D)^{-1} D^T [\dot{e}(t) - A_r e(t)], \quad \sigma_a(0) = \sigma_{a0} \neq 0. \tag{19} \]
where \( k \in \mathbb{R}_+ \). Then, (19) is the negative gradient of
\[ J(\cdot) = \frac{1}{2} \| A^{1/2} (\dot{W}_a^T(t) \sigma(x(t)) + \dot{W}_{u_n}^T u_n(t) + \dot{W}_a^T(t) \sigma_a(t)) \|^2_2, \tag{20} \]
with respect to \( \sigma_a(t) \).

The unknown magnitude of the mismatch term
\[ \dot{W}_a^T(t) \sigma(x(t)) + \dot{W}_{u_n}^T u_n(t) + \dot{W}_a^T(t) \sigma_a(t), \tag{21} \]
in (18) can lead to a large deviation of the state from the reference state during the learning phase of the adaptive controller given by (17). From this standpoint, the proposed artificial basis function allows to shape the system error by suppressing the mismatch term (21) in (18) due to gradient minimization. Therefore, by judiciously adjusting \( k \) in (19), the uncertain dynamical system response and the reference system response can be made close to each other for all time including the the transient phase (note that the requirement on the availability of \( \dot{e}(t) \) in (19) is removed in Theorem 3 of this section). The next theorem presents an appropriate artificial weight update law \( \dot{W}_a(t) \) (and also update laws for \( \dot{W}_\sigma(t) \) and \( \dot{W}_{u_n}(t) \)) to guarantee the asymptotic stability of the closed-loop system error dynamics in (18).

**Theorem 2.** Consider the nonlinear uncertain dynamical system given by (1) subject to Assumption 1, the reference system given by (4), the adaptive feedback control law given by (17) along with the update laws given by (19) and
\[ \begin{align*}
\dot{W}_\sigma(t) &= \gamma_\sigma \begin{bmatrix} \sigma(x(t))e^T(t)PD + \mu \sigma(x(t)) \sigma_\sigma^T(t) \end{bmatrix} W_\sigma(t), \quad W_\sigma(0) = W_{\sigma0}, \tag{22} \\
\dot{W}_{u_n}(t) &= \gamma_{u_n} \begin{bmatrix} u_n(t)e^T(t)PD + \mu u_n(t) \sigma_\sigma^T(t) \end{bmatrix} W_{u_n}(t), \quad W_{u_n}(0) = W_{u_n0}, \tag{23} \\
\dot{W}_a(t) &= \gamma_a \sigma_a(t)e^T(t)PD, \quad W_a(0) = W_{a0} \neq 0, \tag{24}
\end{align*} \]
where \( \gamma_\sigma \in \mathbb{R}_+, \gamma_{u_n} \in \mathbb{R}_+, \gamma_a \in \mathbb{R}_+, \) and \( \mu \in \mathbb{R}_+ \). Then, the solution \( (e(t), \dot{W}_\sigma(t), \dot{W}_{u_n}(t), \dot{W}_a(t), \dot{\sigma}_a(t)) \) of the closed-loop dynamical system is Lyapunov stable for all initial conditions and \( t \in \mathbb{R}_+ \), and \( \lim_{t \to \infty} e(t) = 0 \).

In Theorem 1, (19) presents the update law for the artificial basis function that contains the time derivative of the system error in its right hand side. In practice, it is desired to remove this term from
the update law. Motivating from methods used in [8], the next theorem presents an equivalent form of the update law in (19), but without the time derivative of the system error.

**Theorem 3.** The update law for the artificial basis function given by (19) is identical to

\[
\sigma_a(t) = \sigma_a(0) + k \left[ (\dot{W_a}(t)(D^T D)^{-1}D^T c(t) - \dot{W}_a(0)(D^T D)^{-1}D^T c(0)) \right. \\
- \int_0^t \gamma_a \sigma_a(\tau)e^T(\tau)PD(D^T D)^{-1}D^T e(\tau)d\tau - \int_0^t \dot{W}_a(\tau)(D^T D)^{-1}D^T A_{r\epsilon}(\tau)d\tau \right].
\]

(25)

**IV. Artificial Basis Functions in Adaptive Control with Guaranteed Bounds**

Due to the nature of gradient minimization, a predictable transient performance is obtained by judiciously increasing \( k \) in (25), as this suppresses the cost function given by (20) [1]. However, selection of this gain is problem dependent and may need to be adjusted for each different situation. Therefore, an additional mechanism is necessary to achieve a given design criteria for the adaptive control design in Section III with artificial basis functions employed. To address this problem, we now present a new approach based on the results of [3] to auto-tune the proposed framework of the previous section. Specifically, let \( z \in \mathbb{R}^p \) be a real column vector and

\[
\|z\|_H \triangleq \sqrt{z^THz},
\]

(26)

be a weighted Euclidian norm, where \( H \in \mathbb{R}^{p \times p} \cap S^{p \times p} \). We say \( \phi(\|z\|_H) : \mathbb{R}^p \to \mathbb{R} \), is a restricted potential function (also known as Barrier Lyapunov Function) [3,9–11] defined on

\[
D_\epsilon \triangleq \{\|z\|_H : \|z\|_H \in [0, \epsilon) \},
\]

(27)

if the following statements hold:

i) If \( \|z\|_H = 0 \), then \( \phi(\|z\|_H) = 0 \).

ii) If \( \|z\|_H \in D_\epsilon \) and \( \|z\|_H \neq 0 \), then \( \phi(\|z\|_H) > 0 \).

iii) If \( \|z\|_H \to \epsilon \), then \( \phi(\|z\|_H) \to \infty \).

iv) \( \phi(\|z\|_H) \) is continuously differentiable on \( D_\epsilon \).

v) If \( \|z\|_H \in D_\epsilon \), then \( \phi_d(\|z\|_H) > 0 \), where

\[
\phi_d(\|z\|_H) \triangleq \frac{d\phi(\|z\|_H)}{d\|z\|_H^2}.
\]

(28)

It is given in [3] that a candidate restricted potential function satisfying the above conditions has the form

\[
\phi(\|z\|_H) = \frac{\|z\|_H^2}{\epsilon - \|z\|_H^2}, \quad \|z\|_H \in D_\epsilon,
\]

(29)

which has the derivative \( \phi_d(\|z\|_H) = \frac{\epsilon - \|z\|_H^2}{(\epsilon - \|z\|_H^2)^2} \) with respect to \( \|z\|_H^2 \).

Next, let weight update laws given in (22), (23), and (24) be reformulated as

\[
\dot{W}_a(t) = \gamma_a \left[ \phi_d(\|e(t)\|_P)\sigma_a e(t)P \right] e^T(t)PD + \mu_a \sigma_a e(t)P \dot{W}_a(t), \quad \dot{W}_a(0) = \dot{W}_{a0},
\]

(30)

\[
\dot{W}_{as}(t) = \gamma_a \left[ \phi_d(\|e(t)\|_P)u_a e(t)P \right] e^T(t)PD + \mu_a u_a e(t)P \dot{W}_{as}(t), \quad \dot{W}_{as}(0) = \dot{W}_{as0} \neq 0,
\]

(31)

\[
\dot{W}_a(t) = \gamma_a \phi_d(\|e(t)\|_P)\sigma_a e(t)P \dot{D}, \quad \dot{W}_a(0) = \dot{W}_{a0} \neq 0,
\]

(32)

with \( \gamma_a \phi_d(\|e\|_P), \gamma_a u_a \phi_d(\|e\|_P) \), and \( \gamma_a \phi_d(\|e\|_P) \) being error-dependent learning gains, where \( \gamma_a, \gamma_u_a, \) and \( \gamma_a \in \mathbb{R}_+ \) act as scaling factors, and \( P \in \mathbb{R}_+^{n \times n} \cap S^{n \times n} \) is a solution of the Lyapunov equation given by (11),

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where \( R \in \mathbb{R}^{n \times n} \cap S^{n \times n} \). Unlike standard adaptive control designs, where a constant high-gain is used all the time to achieve a given system performance, these state-dependent learning gains may get large only when it is absolutely necessary (e.g., when the controlled system is about to violate a given design constraint on error dynamics) to strictly enforce a user-defined bound on the error dynamics. The following theorem presents the main result of this paper.

**Theorem 4.** Consider the nonlinear uncertain dynamical system given by (1) subject to Assumption 1, the reference system given by (4), the feedback control law given by (17) along with the update laws given by (19), (30), (31), and (32). If \( \|e_0\| \leq \epsilon \) (where \( \epsilon \) is an arbitrary design performance parameter), then the solution \((e(t), \hat{W}_a(t), \hat{W}_u(t), \sigma_a(t))\) of the closed-loop dynamical system is Lyapunov stable for all initial conditions and \( t \in \mathbb{R}_+ \), and \( \lim_{t \to \infty} e(t) = 0 \) with the guaranteed \textit{a priori} known transient performance bound given by

\[
\|e(t)\|_p < \epsilon, \quad t \in \mathbb{R}_+. \tag{33}
\]

**Proof.** Define \( V : D_e \times \mathbb{R}^{\dim(\hat{W}_a)} \times \mathbb{R}^{\dim(\hat{W}_u)} \times \mathbb{R}^{\dim(\sigma_a)} \times \mathbb{R}^{\dim(\sigma_u)} \to \mathbb{R}_+ \) as

\[
V(e, \hat{W}_a, \hat{W}_u, \sigma_a) = \phi(\|e\|_p) + \gamma_{\sigma}^{-1} \text{tr} \left( \hat{W}_a \Lambda^{1/2} \right)^T \left( \hat{W}_a \Lambda^{1/2} \right) + \gamma_{\sigma}^{-1} \text{tr} \left( \hat{W}_u \Lambda^{1/2} \right)^T \left( \hat{W}_u \Lambda^{1/2} \right)
\]

\[+ \gamma_{\sigma}^{-1} \text{tr} \left( \hat{W}_a \Lambda^{1/2} \right)^T \left( \hat{W}_a \Lambda^{1/2} \right) + \mu k^{-1} \sigma_a^T \sigma_a, \tag{34}\]

where

\[
D_e \triangleq \{ e(t) : \|e(t)\|_p < \epsilon \}, \tag{35}\]

and \( \dim(\hat{W}_a), \dim(\hat{W}_u), \dim(\hat{W}_\sigma), \) and \( \dim(\sigma_u) \) denote the dimensions of \( \hat{W}_a, \hat{W}_u, \hat{W}_\sigma, \) and \( \sigma_u \) respectively. For any \( \xi > 0 \), let

\[
\Omega \triangleq \{ (e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) \in \mathcal{H} : V(e(t), \hat{W}_a(t), \hat{W}_u(t), \sigma_a(t)) \leq \xi \}, \tag{36}\]

denote the level sets of \( V(e, \hat{W}_a, \hat{W}_u, \hat{W}_\sigma, \sigma_a) \), where \( \mathcal{H} = D_e \times \mathbb{R}^{\dim(\hat{W}_a)} \times \mathbb{R}^{\dim(\hat{W}_u)} \times \mathbb{R}^{\dim(\hat{W}_\sigma)} \times \mathbb{R}^{\dim(\sigma_u)} \). Note that \( V(0,0,0,0,0) = 0 \) and \( V(e, \hat{W}_a, \hat{W}_u, \hat{W}_\sigma, \sigma_a) > 0 \) for all admissible \((e, \hat{W}_a, \hat{W}_u, \hat{W}_\sigma, \sigma_a) \neq (0,0,0,0,0) \). Furthermore, considering

\[
\frac{d\phi(\|e(t)\|_p)}{dt} = 2\phi_d(\|e(t)\|_p) e^T(t) P e(t), \tag{37}\]

it follows from (34) that

\[
\dot{V}(e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) \leq -\phi_d(\|e(t)\|_p) e^T(t) Re(t) \leq 0, \tag{38}\]

for \((e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) \in \mathcal{H} \) and \( t \in \mathbb{R}_+ \).

Next, we follow a similar approach to [12,13] for concluding our proof. The level sets of \( V(e, \hat{W}_a, \hat{W}_u, \hat{W}_\sigma, \sigma_a) \) and \( \Omega \) are compact and invariant. In particular, the set \( \Omega \) for \( \xi > 0 \) is closed by the continuity of \( V(e, \hat{W}_a, \hat{W}_u, \hat{W}_\sigma, \sigma_a) \) for \((e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) \in \mathcal{H} \). Let \( Q \) be the set of all points in \( \Omega \) such that \( \dot{V}(e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) = 0 \),

\[
Q \triangleq \{ (e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) \in \mathcal{H} : \dot{V}(e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) = 0 \}
\]

\[
= \{ (e(t), \hat{W}_a(t), \hat{W}_u(t), \hat{W}_\sigma(t), \sigma_a(t)) \in \mathcal{H} : e = 0 \}. \tag{39}\]

It now follows that all solutions approach the largest invariant set \( R \) in \( Q \). This concludes the proofs since \( R \) is composed of all points in which

\[
e(t) = 0, \tag{40}\]

\[
D \Lambda [\hat{W}_u^T(t) \sigma(x(t)) + \hat{W}_u^T(t) u_n(t) + \hat{W}_a^T(t) \sigma_a(t)] = 0. \tag{41}\]
V. Illustrative Example

In order to illustrate the designable performance bounds, consider the nonlinear dynamical system representing a controlled wing rock dynamics model given by

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t), \quad x_1(0) = 0, \\
\dot{x}_2(t) &= \Lambda u(t) + \delta(x(t)), \quad x_2(0) = 0,
\end{align*}
\]

where \(x_1\) represents the roll angle in radians and \(x_2\) represents the roll rate in radians per second. In (43), \(\delta(x)\) represents an uncertainty of the form \(\delta(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 |x_1| x_2 + \alpha_4 |x_2| x_2 + \alpha_5 x_1^3\), where \(\alpha_i, i = 1, \ldots, 5\), are unknown parameters that are derived from the aircraft aerodynamic coefficients. For our numerical example, we set \(\alpha_1 = 0.1414, \alpha_2 = 0.5504, \alpha_3 = -0.0624, \alpha_4 = 0.0095, \alpha_5 = 0.0215\), and \(\Lambda = 0.5\). We choose \(K_1 = [-0.16, -0.57]\) and \(K_2 = 0.16\) for the nominal controller design that yields to a reference system with a natural frequency of \(\omega_n = 0.40\) rad/s and a damping ratio \(\zeta = 0.707\). For the adaptive controller design based on artificial basis functions given by (5), (6), (17), (22), (23), (24), and (25), \(\sigma(x) = [x_1, x_2, |x_1| x_2, |x_2| x_2, x_1^3]^T\) is used for the basis function and we set \(R = I_2\) and \(q = 1\) is chosen implying we used a one dimensional artificial basis function. For the proposed adaptive controller with certified bounds given by (5), (6), (17), (25), (30), (31), and (32), we use the same basis function, \(R = I_2\), \(q = 1\), and set \(\epsilon = 0.1\) rad (corresponding to 5.57 degrees) in (29).

Figure 1 shows the system response under the adaptive control using artificial basis functions behaves smoothly and the system state tracks the model well; however, there is a peak in \(t \in [5, 15]\) seconds. Figure 2 shows \(||e(t)||_\rho\) violates \(\epsilon\) since the learning gain of this design is not adjusted, i.e., \(\phi_1(||e(t)||_\rho) = 1\). By applying the proposed controller utilizing the error-dependent learning gains in the update laws, we are able to enforce a constraint in the error. Figure 3 shows a reduction in the peak previously seen in Figure 1, which is due to the enforced error constraint. As seen in Figure 4, the system error stays within the enforced bound of \(\epsilon\) due to the change in the error-dependent learning gain. In Figures 5 and 6, we inserted significant measurement noise into the system. It is evident the system maintains desirable performance while the system error remains bounded by the enforced constraint.

![Figure 1. Artificial basis function adaptive control performance with (5), (6), (17), (22), (23), (24), and (25) for a given square-wave tracking command (\(\hat{V}_{ad} = 0.1, \sigma_{ad} = 0.1, \gamma_v = 1, \gamma_a_u = 1, \gamma_a = 1, k = 10, \mu = 1\)).](image-url)
Figure 2. System error bounds and adaptation gain for Figure 1.

Figure 3. Proposed adaptive control performance with (5), (6), (17), (25), (30), (31), and (32) for a given square-wave tracking command ($\hat{W}_a = 0.1$, $\sigma_a = 0.1$, $\gamma_a = 1$, $\gamma_{u_a} = 1$, $\gamma_a = 1$, $k = 10$, $\mu = 1$, $\epsilon = 0.1$).

VI. Conclusion

To contribute to the previous studies in adaptive control theory, we investigated a new artificial basis functions-based adaptation architecture with auto-tuning capability, where the proposed framework yields to a (user-defined) predictable transient performance. Future research will include rigorous characterization of robustness with respect to measurement noise and will also investigate adjustment of the gain $k$ used in (25) and we will study how it compares to the adjustment of the learning rates in (30), (31), and (32).
Figure 4. System error bounds and adaptation gain for Figure 3.

Figure 5. Proposed adaptive control performance with (5), (6), (17), (25), (30), (31), and (32) for a given square-wave tracking command with noise ($\hat{W}_{a0} = 0.1$, $\sigma_{a0} = 0.1$, $\gamma_{a} = 1$, $\gamma_{un} = 1$, $\gamma_{u} = 1$, $k = 10$, $\mu = 1$, $\epsilon = 0.1$).

References


Figure 6. System error bounds and adaptation gain for Figure 5.


