Model Reference Adaptive Control in the Presence of High-Order Actuator Dynamics

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Abstract—We recently proposed a linear matrix inequalities-based hedging approach to compute stability limits of adaptive controllers in the presence of first-order actuator dynamics. Specifically, our approach modifies the ideal reference model dynamics using the hedging method to allow correct adaptation, which is not affected by the presence of actuator dynamics, and then analyzes the stability of this modified reference model coupled with the first-order actuator dynamics using linear matrix inequalities — for computing the fundamental stability interplay between the bandwidth of actuator dynamics and the allowable system uncertainties. This paper generalizes this framework to high-order (linear time-invariant) actuator dynamics and discuss the distance between the uncertain dynamical system and the ideal (i.e., unmodified) reference model dynamics. An illustrative numerical example is provided to demonstrate the efficacy of the proposed approach in computing stability limits of adaptive controllers.

I. INTRODUCTION

If the actuator dynamics do not have sufficiently high bandwidth, then their presence in the feedback control loop cannot be neglected. In this case, stability verification steps must be taken to show the fundamental stability interplay between the bandwidth of actuator dynamics and the allowable system uncertainties. As compared with a few contributions to the stability of adaptive controllers in the presence of actuator dynamics, the authors of [1], [2] propose a novel hedging method, where this method modifies the ideal reference model dynamics to allow correct adaptation, which is not affected by the presence of actuators. Yet, until our recent work [3], [4], it has not been analyzed that this modification to the ideal reference model dynamics does not yield to unbounded reference model responses in the presence of actuator dynamics.

Specifically, a linear matrix inequalities (LMIs)-based hedging method is proposed in [3], [4] to compute stability limits of adaptive controllers in the presence of first-order actuator dynamics. This approach utilizes the hedging method to correct adaptation and then we analyze the stability of the modified reference model coupled with the first-order actuator dynamics using LMIs — for computing the fundamental stability interplay between the bandwidth of actuator dynamics and the allowable system uncertainties. The contribution of this paper is to generalize this framework to high-order (linear time-invariant) actuator dynamics and discuss the distance between the uncertain dynamical system and the ideal (i.e., unmodified) reference model dynamics. An illustrative numerical example is provided to demonstrate the efficacy of the proposed approach in computing stability limits of adaptive controllers.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (respectively, $\mathbb{R}_+$) denotes the set of positive (respectively, nonnegative) real numbers, $\mathbb{R}^{n \times n}_+$ (respectively, $\mathbb{R}^{n \times n}_+$) denotes the set of $n \times n$ positive-definite (respectively, nonnegative-definite) real matrices, $\mathbb{S}^{n \times n}$ denotes the set of $n \times n$ symmetric real matrices, $\mathbb{D}^{n \times n}$ denotes the set of $n \times n$ real matrices with diagonal scalar entries, $(\cdot)^T$ denotes the transpose operator, $(\cdot)^{-1}$ denotes the inverse operator, $\text{tr}(\cdot)$ denotes the trace operator, $\|\cdot\|_2$ denotes the Euclidian norm, $\|\cdot\|_F$ denotes the Frobenius matrix norm, $[A]_{ij}$ denotes the $ij$-th entry of the real matrix $A \in \mathbb{R}^{n \times m}$, $\lambda_{\min}(A)$ (respectively, $\lambda_{\max}(A)$) denotes the minimum (respectively, maximum) eigenvalue of the real matrix $A \in \mathbb{R}^{n \times n}$, and $\Delta^*$ denotes the equality by definition.

II. MATHEMATICAL PRELIMINARIES

Necessary mathematical preliminaries are introduced in this section. Specifically, we begin with the following definition of the projection operator.

Definition 1. A convex hypercube in $\mathbb{R}^n$ is defined by

$$
\Omega = \{ \theta \in \mathbb{R}^n : (\theta_{i}^{\min} \leq \theta_i \leq \theta_{i}^{\max})_{i=1,2,\ldots,n} \},
$$

(1)

where $(\theta_{i}^{\min}, \theta_{i}^{\max})$ represent the minimum and maximum bounds for the $i$-th component of the $n$-dimensional parameter vector $\theta$. In addition, for a sufficiently small positive constant $\epsilon$, a second hypercube is defined by
\[ \Omega_{e} = \{ \theta \in \mathbb{R}^n : (\theta_{i}^{\text{min}} + \epsilon \leq \theta_{i} \leq \theta_{i}^{\text{max}} - \epsilon)_{i=1,2,\ldots,n} \}, \]  

(2)

where \( \Omega_{e} \subset \Omega \). Then, the projection operator \( \text{Proj} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is defined component-wise by

\[
\text{Proj}(\theta, y) \triangleq \begin{cases} 
\left( \frac{\theta_{i}^{\text{max}} - \theta_{i}}{\epsilon} \right) y_{i}, & \text{if } \theta_{i} > \theta_{i}^{\text{max}} - \epsilon \\
\left( \frac{\theta_{i}^{\text{min}} - \theta_{i}}{\epsilon} \right) y_{i}, & \text{if } \theta_{i} < \theta_{i}^{\text{min}} + \epsilon \\
y_{i}, & \text{otherwise}
\end{cases}
\]

(3)

where \( y \in \mathbb{R}^n \) [5].

It follows from Definition 1 that

\[
(\theta - \theta^{*})^T (\text{Proj}(\theta, y) - y) \leq 0, \quad \theta^{*} \in \mathbb{R}^n,
\]

(4)

holds [5, 6]. We use a generalization of this definition to matrices as

\[
\text{Proj}_{m}(\Theta, Y) = (\text{Proj}(\text{col}_{1}(\Theta), \text{col}_{1}(Y)), \ldots, \text{Proj}(\text{col}_{m}(\Theta), \text{col}_{m}(Y))),
\]

(5)

where \( \Theta \in \mathbb{R}^{n \times m}, Y \in \mathbb{R}^{n \times m} \), and \( \text{col}_{i}(\cdot) \) denotes the \( i \)-th column operator. In this case, for a given \( \Theta^{*} \in \mathbb{R}^{n \times m} \), it follows that

\[
\text{tr} \left[ (\Theta - \Theta^{*})^T (\text{Proj}_{m}(\Theta, Y) - Y) \right] = \sum_{i=1}^{m} \text{col}_{i}(\Theta - \Theta^{*})^T (\text{Proj}(\text{col}_{i}(\Theta), \text{col}_{i}(Y)) - \text{col}_{i}(Y)) \leq 0,
\]

(6)

holds.

We now briefly overview the standard model reference control problem. Specifically, consider the uncertain dynamical system given by

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,
\]

(7)

where \( x(t) \in \mathbb{R}^n \) is the state vector available for feedback, \( u(t) \in \mathbb{R}^m \) is the control input restricted to the class of admissible controls consisting of measurable functions, \( A \in \mathbb{R}^{n \times n} \) is an unknown system matrix, \( B \in \mathbb{R}^{n \times m} \) is a known input matrix, and the pair \((A, B)\) is controllable.

Next, consider the reference model capturing a desired, ideal closed-loop dynamical system performance given by

\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_0,
\]

(8)

where \( x_r(t) \in \mathbb{R}^n \) is the reference state vector, \( c(t) \in \mathbb{R}^m \) is a given uniformly continuous bounded command, \( A_r \in \mathbb{R}^{n \times n} \) is the Hurwitz reference model matrix, and \( B_r \in \mathbb{R}^{n \times m} \) is the command input matrix. The objective of the model reference adaptive control problem is to construct an adaptive feedback control law \( u(t) \) such that the state vector \( x(t) \) asymptotically follows the reference state vector \( x_r(t) \). We now make the following assumption, which is also standard in the model reference adaptive control literature and is known as the matching condition [5].

\textbf{Assumption 1}. There exists an unknown matrix \( K_1 \in \mathbb{R}^{n \times n} \) and a known matrix \( K_2 \in \mathbb{R}^{m \times m} \) such that

\[
A_r = A - BK_1,
\]

(9)

\[
B_r = BK_2,
\]

(10)

hold.

It follows from Assumption 1 that (7) can be written as

\[
\dot{x}(t) = A_r x_r(t) + B_r c(t) + B [u(t) + W_1(t)x(t) - K_2 c(t)],
\]

(11)

where

\[
W_1 \triangleq K_1^T \in \mathbb{R}^{n \times m},
\]

(12)

is unknown. Now, let the adaptive feedback control law be given by

\[
u(t) = -\hat{W}_1^T(t)x(t) + K_2 c(t),
\]

(13)

where \( \hat{W}_1(t) \in \mathbb{R}^{n \times m} \) is the estimate of \( W_1 \) satisfying the weight update law

\[
\dot{\hat{W}}_1(t) = \gamma_1 \text{Proj}_m \left[ \hat{W}_1(t), x(t) e^T(t) P B \right],
\]

(14)

with \( \gamma_1 \in \mathbb{R}_+ \) being the learning rate,

\[
e(t) \triangleq x(t) - x_r(t),
\]

(15)

being the system error state vector, and \( P \in \mathbb{R}^{+ \times n} \cap \mathbb{S}^{n \times n} \) being the solution of the Lyapunov equation given by

\[
0 = A^T P + PA_r + R,
\]

(16)

\( R \in \mathbb{R}_+^{n \times n} \cap \mathbb{S}^{n \times n} \).

Note that since \( A_r \) is Hurwitz, it follows from the converse Lyapunov theory that there exists a unique \( P \) satisfying (16) for a given \( R \). In addition, the projection bounds are defined such that

\[
[\hat{W}_1(t)]_{ij} \leq \hat{W}_{1, \text{max}, i+(j-1)n},
\]

(17)

for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \), where

\[
\hat{W}_{1, \text{max}, i+(j-1)n} \in \mathbb{R}_+,
\]

(18)

denotes (symmetric) element-wise projection bounds. Note that the results of this paper can be readily applied to the case when asymmetric projection bounds are considered.

Now, using (13) in (11) along with (8), the system error dynamics can be written as

\[
\dot{e}(t) = A_r e(t) - B \hat{W}_1^T(t)x(t), \quad e(0) = e_0,
\]

(19)

where

\[
\hat{W}_1(t) \triangleq \hat{W}_1(t) - W_1 \in \mathbb{R}^{n \times m}.
\]

(20)

Note that the weight update law given by (14) can be derived using Lyapunov analysis by considering the Lyapunov
function candidate given by (see, for example, [5])
\[ \mathcal{V}(e, \dot{W}_1) = e^T P e + \gamma_1^{-1} \text{tr } \dot{W}_1^T W_1. \] (21)

Note that \( \mathcal{V}(0,0) = 0 \) and \( \mathcal{V}(e, \dot{W}_1) > 0 \) for all \( (e, \dot{W}_1) \neq (0,0) \). Now, differentiating (21) yields
\[ \dot{\mathcal{V}}(e(t), \dot{W}_1(t)) \leq -e^T(t) R e(t) \leq 0, \] (22)
which guarantees that the system error state vector \( e(t) \) and the weight error \( \dot{W}_1(t) \) are Lyapunov stable, and hence, are bounded for all \( t \in \mathbb{R}_+ \). Since \( x(t) \) is bounded for all \( t \in \mathbb{R}_+ \), it follows from (19) that \( \dot{e}(t) \) is bounded, and hence, \( \dot{\mathcal{V}}(e(t), \dot{W}_1(t)) \) is bounded for all \( t \in \mathbb{R}_+ \). It then follows from Barbalat’s lemma that
\[ \lim_{t \to \infty} \dot{\mathcal{V}}(e(t), \dot{W}_1(t)) = 0, \] (23)
which consequently shows that
\[ \lim_{t \to \infty} e(t) = 0. \] (24)

The above discussion highlights that the adaptive control formulation introduced in this section has the capability to suppress the effect of system uncertainties to achieve desirable tracking performance specifications. Yet, it does not provide guarantees in the presence actuator dynamics that appear in any practical application of adaptive controllers.

### III. Adaptive Control in the Presence of High-Order Actuator Dynamics

For the model reference adaptive control framework introduced in the previous section, we now introduce the actuator dynamics problem. Specifically, consider
\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \] (25)
where \( u(t) \in \mathbb{R}^m \) is the actuator output of the actuator dynamics \( G_A \) given by
\[ \dot{x}_c(t) = Fx_c(t) + Gu(t), \quad x_c(0) = x_c_0, \] (26)
with \( x_c(t) \in \mathbb{R}^p \) being the actuator state vector, \( G \in \mathbb{R}^{p \times m} \) being the actuator input matrix, \( F \in \mathbb{R}^{m \times p} \) being the actuator output matrix, and \( F \in \mathbb{R}^{p \times p} \) being a Hurwitz matrix in Jordan form (without loss of generality) such that there exists \( S \in \mathbb{R}^{p \times p} \cap \mathbb{S} \) that satisfies
\[ 0 = F^T S + SF + I. \] (27)
In addition, we introduce the following assumption.

**Assumption 2.** The static gain of the actuator dynamics given by (26) is unity; that is,
\[ -HF^{-1}G = I, \] (28)
and the algebraic multiplicity of \( F \) is equal to its geometric multiplicity.

By adding and subtracting \( Bu(t) \) and using Assumption 1, (25) can be rewritten as
\[ \dot{x}(t) = Ax(t) + Bu(t) + W_1^T(t)x(t) - K_2 \dot{c}(t) + B[u(t) - \dot{u}(t)]. \] (29)

Now, based on the hedging approach [1], [2], we consider the modified reference model dynamics given by
\[ \dot{x}_c = A_r \dot{x}_c(t) + B_r c(t) + B \dot{u}(t), \quad x_c(0) = x_c_0, \] (30)
such that with the adaptive feedback control law given by (13) and (14), the system error dynamics follows from (29) and (30) as
\[ \dot{e}(t) = A_r \dot{c}(t) - B \dot{W}_1^T(t)x(t), \quad e(0) = e_0. \] (31)
Notice that (31) is identical to the system error dynamics given by (19) due to the fact that the hedging signal
\[ B \dot{u}(t), \] (32)
is introduced to the ideal reference model dynamics. For the next result, we introduce the following assumption.

**Assumption 3.** The matrix given by
\[ A(W_1(t), G_A) = \begin{bmatrix} A_r + BW_1^T(t) B & BH \\ -GW_1^2(t) & F \end{bmatrix}, \] (33)
is quadratically stable.

**Theorem 1.** Consider the uncertain dynamical system given by (25) subject to Assumption 1, the reference model given by (30), the actuator dynamics given by (26) subject to Assumption 2, and the adaptive feedback control law given by (13) along with the update law (14). If Assumption 3 holds, then the solution \((e(t), \dot{W}_1(t), x_c(t), v(t))\) of the closed-loop dynamical system are bounded and
\[ \lim_{t \to \infty} e(t) = 0. \] (34)
In addition, the system error dynamics satisfy the transient performance bound given by
\[ ||e(t)||_{\infty} \leq \left( \frac{1}{\lambda_{\min}(P)} \left( \frac{\lambda_{\max}(P)}{2} ||e(0)||_{2} \right) \right)^{1/2} + \gamma_1^{-1}||\dot{W}_1(0)||^2_F. \] (35)

**Proof.** The result follows from the consideration of the Lyapunov function candidate given by (21) and the boundedness analysis of the pair \((x_r(t), x_c(t))\) based on Assumption 3, which it is omitted due to page limitations.

Note that quadratic stability of the matrix given by (33) plays a central role for the results in Theorem 1, where it reveals the fundamental stability interplay between the allowable system uncertainties (through the selection of the projection operator bounds) and the bandwidth of the high-order actuator dynamics.

We now utilize LMIs to satisfy the quadratic stability of (33) by following a similar procedure documented in our
recent works [3], [4]. For this purpose, let \( \overline{W}_{i_1, \ldots, i_l} \in \mathbb{R}^{n \times m} \) be defined as
\[
\begin{bmatrix}
(1)^{i_1} \overline{W}_{\text{max}, 1} & (1)^{i_2+n} \overline{W}_{\text{max}, 1+n} \\
(1)^{i_2} \overline{W}_{\text{max}, 2} & (1)^{i_2+n} \overline{W}_{\text{max}, 2+n} \\
\vdots & \vdots \\
(1)^{i_l} \overline{W}_{\text{max}, n} & (1)^{i_2+n} \overline{W}_{\text{max}, 2+n}
\end{bmatrix}
\]
\[\vdots \]
\[
\begin{bmatrix}
(1)^{i_1+(m-1)n} \overline{W}_{\text{max}, 1+(m-1)n} \\
(1)^{i_2+(m-1)n} \overline{W}_{\text{max}, 2+(m-1)n} \\
\vdots \\
(1)^{i_m} \overline{W}_{\text{max}, mn}
\end{bmatrix}, \tag{36}
\]
where \( i_l \in \{1, 2\}, l \in \{1, \ldots, mn\} \), such that \( \overline{W}_{i_1, \ldots, i_l} \) represents the corners of the hypercube defining the maximum variation of \( \overline{W}_i(t) \). Utilizing the results in [7], [8], if
\[
A_{i_1, \ldots, i_l} = \begin{bmatrix}
A_r + BW_{i_1, \ldots, i_l}^T B H \\
-GW_{i_1, \ldots, i_l}^T F
\end{bmatrix}, \tag{37}
\]
satisfies the matrix inequality
\[
A_{i_1, \ldots, i_l}^T \mathcal{P} + \mathcal{P} A_{i_1, \ldots, i_l} < 0, \quad \mathcal{P} = \mathcal{P}^T > 0, \tag{38}
\]
for all permutations of \( \overline{W}_{i_1, \ldots, i_l} \), then (33) is quadratically stable. Since it was shown in the proof of Lemma 1 that (33) is quadratically stable for large values of \( k \), we can cast (38) as a convex optimization problem and solve it using LMIs.

IV. DEVIATION ANALYSIS FROM THE IDEAL REFERENCE MODEL DYNAMICS

The previous section shows that the distance between the uncertain dynamical system given by (25) and the modified reference model given by (30) asymptotically vanishes. In this section, we analyze the distance between the uncertain dynamical system and the ideal (i.e., unmodified) reference model given by
\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \tag{39}
\]
where \( x_r(t) \in \mathbb{R}^n \) is the ideal reference state vector.

We begin by defining
\[
e_r(t) \triangleq x_r(t) - x_{r_i}(t), \tag{40}
\]
as the error between the modified reference model given by (30) and the ideal reference model given by (39). Note that
\[
\|x(t) - x_r(t)\|_{\infty} \leq \|e(t) + e_r(t)\|_{\infty} \leq \|e(t)\|_{\infty} + \|e_r(t)\|_{\infty}, \tag{41}
\]
which implies that by making the bounds on the both error signals (i.e., \( \|e(t)\|_{\infty} \) and \( \|e_r(t)\|_{\infty} \)) small, the distance between the uncertain dynamical system and the ideal reference model becomes small for all time.

The next theorem proves that \( \|e_r(t)\|_{\infty} \) can be made small if the actuator dynamics are fast. For the following result, we first note that the actuator dynamics \( \tilde{G}_A \), which corresponds to
\[
\tilde{F} \triangleq k F_0, \tag{42}
\]
\[
\tilde{H} \triangleq k H_0, \tag{43}
\]
\[
\tilde{G} \triangleq G_0, \tag{44}
\]
can be related to the actual actuator dynamics \( G_A \) as
\[
F \triangleq l \tilde{F}, \tag{45}
\]
\[
H \triangleq l \tilde{H}, \tag{46}
\]
\[
G \triangleq \tilde{G}, \tag{47}
\]
such that Assumption 2 holds, where \( l \in \mathbb{R}_+ \) and \( l > 1 \). In addition, as a result of Assumption 2 and (27), we can also write
\[
S = l^{-1} \text{diag} \left( [l^{-1} \tilde{F}_{11}, l^{-1} \tilde{F}_{22}, \ldots, l^{-1} \tilde{F}_{pp}] \right) / 2 \triangleq l^{-1} \tilde{S}, \tag{48}
\]
with
\[
0 = \tilde{F}^T \tilde{S} + \tilde{S} \tilde{F} + I, \tag{49}
\]
and \( \tilde{S} \in \mathbb{R}^{l \times p} \). Finally, we define
\[
d_1 \triangleq \alpha \|\tilde{S} \tilde{G}_r\|_{\text{if}, \omega^*_1}, \tag{50}
\]
\[
d_2 \triangleq \alpha^2 \|\tilde{S} \tilde{G}_r^2\|_{\text{if}, \omega^*_2}, \tag{51}
\]
\[
\rho \triangleq \sqrt{\frac{\lambda_{\text{max}}(P)}{\lambda_{\text{min}}(P)}}, \tag{52}
\]
where \( \omega^*, \omega^*_1, \eta \in \mathbb{R}_+ \), and \( \eta > 1 \).

Theorem 2. Consider the modified reference model given by (30), the ideal reference model given by (39), the actuator dynamics given by (26) subject to Assumption 2, the adaptive feedback control law given by (13). If Assumption 3 holds, then
\[
\|e_r(t)\|_2 \to 0 \quad \text{as} \quad \lambda_{\text{min}}(\tilde{F} - F) \to \infty, \tag{53}
\]
and is guaranteed to shrink at the rate
\[
\|e_r(t)\|_{\infty} \leq \rho \left( \frac{d_1^2}{l^2 (\eta - (1 - l^{-1}))} \right) \left( \eta - d_2 (1 - l^{-1}) \right) + \frac{4d_2^2}{l^2 (\eta - (1 - l^{-1}))} \frac{1}{2}. \tag{54}
\]

Proof. The result utilizes the arguments presented in [9], which it is omitted due to page limitations.

Under a realistic assumption that \( e(0) \) can be chosen to be small, Theorem 1 shows that \( \|e(t)\|_{\infty} \) can be made small by judiciously increasing the learning gain \( \gamma_1 \). In addition, Theorem 2 shows that \( \|e_r(t)\|_{\infty} \) becomes small as the actuator dynamics becomes fast. Hence, it follows that
the upper bound (41) on the distance between the uncertain dynamical system and the ideal reference model can be made small by judiciously increasing the learning gain $\gamma_1$ as well as the bandwidth of the actuator dynamics.

V. ILLUSTRATIVE EXAMPLE

In order to illustrate the proposed adaptive control architecture with actuator dynamics, we consider the second-order system given by

$$
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} v(t),
$$

with zero initial conditions, and let $x_1(t)$ represent the angle in radians and $x_2(t)$ represent the angular rate of change in radians per second. For the high order actuator dynamics, we consider a single channel second order actuator for the control input such that

$$
F = \begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta\omega_n
\end{bmatrix}, \quad G = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad H = \begin{bmatrix}
\omega_n^2 & 0
\end{bmatrix},
$$

where it is noted that $F$ can be transformed into Jordan form such that Assumption 2 is satisfied. In addition, we use a filtered tracking command $c(t)$ and select a reference model with zero initial conditions, a natural frequency of $\omega_n = 0.7$ rad/s, and a damping ratio $\zeta_r = 0.707$, which yields

$$
A_r = \begin{bmatrix}
0 & 1 \\
-0.49 & -0.9898
\end{bmatrix}, \quad B_r = \begin{bmatrix}
0 \\
0.49
\end{bmatrix}.
$$

We set $R = I_2$ from (16) for the proposed adaptive controller designs and using the rectangular projection operator, the bounds on the uncertainty are set element-wise such that $\|\hat{W}_1(t)\|_{1,1} \leq 1.1$ and $\|\hat{W}_1(t)\|_{2,1} \leq 1.6$. Then using the bounds on $\hat{W}_1(t)$ in the LMI analysis highlighted in Section III, the feasible region of allowable actuator dynamics is calculated.

Figure 1 shows the feasible region of allowable actuator dynamics which is given by the $\omega_n$ and $\zeta$ values for the actuator dynamics. Note that Figure 1 provides both the LMI calculated feasible limit as well as the feasible limit provided by the simulation results. Due to space restrictions, we select two points to simulate the proposed controller performance as seen in Figures 2 and 3. Since the feasible boundary corresponds to calculated minimum feasible $\omega_n$ and $\zeta$ values for the actuator dynamics, it is expected that the system performances are guaranteed to be bounded for actuator dynamics at points greater than and equal to the calculated feasible boundary. This can be seen in Figure 2 when the actuator dynamics are at the minimum point

$$(\zeta, \omega_n) = (0.55, 2.98),$$

which is located on the feasible boundary. In Figure 3, we let the actuator dynamics be outside the calculated feasible region to show that the closed-loop system remains bounded until the actuator dynamics reach a value of

$$(\zeta, \omega_n) = (0.55, 2.19).$$

This is consistent with the presented theory, as we provide a (conservative) upper bound on the allowable actuator dynamics such that the closed-loop system remains bounded.

VI. CONCLUSION

To contribute to the previous studies in adaptive control of uncertain dynamical systems in the presence of high-order (linear time-invariant) actuator dynamics, we presented an LMI-based hedging approach for computing the fundamental

Fig. 1. LMI calculated feasible region for actuator dynamics.
stability interplay between the bandwidth of actuator dynamics and the allowable system uncertainties. Specifically, the proposed approach modifies the ideal reference model dynamics using the hedging method to allow correct adaptation, which is not affected by the presence of actuator dynamics. We analyzed the stability of this modified reference model coupled with the actuator dynamics using tools and methods from Lyapunov stability, matrix mathematics, and LMIs. In addition, the distance between the uncertain dynamical system and the ideal (i.e., unmodified) reference model dynamics were also analyzed and it was remarked that this distance either can be made small by increasing the learning gain and the bandwidth of the actuator dynamics. An illustrative numerical example demonstrated the efficacy of the proposed approach in computing stability limits of adaptive controllers in the presence of high-order actuator dynamics.

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