Computing Actuator Bandwidth Limits for Adaptive Control

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In recent studies, an LMI-based hedging framework has been developed and demonstrated for model reference adaptive controllers in order to achieve command following in the presence of actuator dynamics. In this paper, we generalize this framework for dynamical systems subject to uncertainty in the control effectiveness and provide a detailed stability analysis of the proposed approach. An illustrative numerical example is further provided to demonstrate the efficacy of the proposed framework.

I. Introduction

The presence of actuator dynamics can seriously limit the stability and achievable performance of adaptive controllers. If the actuator dynamics have sufficiently high bandwidth, then their presence can be neglected in the design of model reference adaptive controllers. However, if the actuator dynamics do not have high bandwidth and/or for safety-critical applications of model reference adaptive control theory, stability verification steps must be taken in order to show the allowable bandwidth for actuator dynamics such that adaptive controllers work correctly.

Authors of [1–5] present notable contributions that allows the design of model reference adaptive controllers in the presence of actuator dynamics. Specifically, [1–3] present direct approaches to this problem such that the resulting closed-loop dynamical systems, which are explicitly affected by the presence of actuator dynamics, are analyzed. The framework presented in [4], while not explicitly applied to the problem of actuator dynamics, it provides a novel approach using linear matrix inequalities to compute a minimum filter bandwidth and guarantee system stability. A similar analysis can then be applied to the problem of actuator dynamics such that a minimum actuator bandwidth can be calculated while ensuring stability of the system. A novel hedging approach is proposed in [5], where this approach enables adaptive controllers to be designed
such that their adaptation performances (i.e., their learning performances of system uncertainties) are not affected by the presence of actuator dynamics. This is accomplished by modifying the ideal reference model dynamics with the hedge signal such that correct adaptation is achieved even in the presence of actuator dynamics. However, it has not been analyzed that this modification to the ideal reference model dynamics does not yield to unbounded reference model trajectories in the presence of actuator dynamics.

In the recent prior work of the authors in [6], an LMI-based hedging framework has been developed and demonstrated for model reference adaptive controllers in order to achieve command following in the presence of actuator dynamics. Unlike the results in [5], this framework characterizes the fundamental interplay between the allowable system uncertainties and the bandwidth of the actuator dynamics, and shows a systematic methodology to achieve bounded reference model trajectories while applying the hedging approach. In addition, the proposed framework is also applicable to situations where the output of the actuator dynamics is unknown. The contribution of this paper is to generalize the results in [6] to dynamical systems subject to uncertainty in the control effectiveness and provide a detailed stability analysis of the proposed approach. Although this paper considers a particular model reference adaptive control formulation to present its main results, our results can be used in a complimentary way with many other approaches to adaptive control (including but not limited to [7–12]).

The organization of this paper is as follows. Section 2 covers the notation used in this paper and necessary mathematical preliminaries. Section 3 presents the proposed LMI-based hedging approach for uncertain dynamical systems with uncertain control effectiveness subject to actuator dynamics with known and unknown outputs. An illustrative numerical example is provided in Section 4 to demonstrate the efficacy of the proposed approach to model reference adaptive control and conclusions are summarized in Section 5.

II. Notation and Mathematical Preliminaries

We briefly begin by providing the notation used throughout this paper. Specifically, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_+^n \)) denotes the set of positive (resp. non-negative-definite) real numbers, \( \mathbb{R}_+^{n \times n} \) (resp. \( \mathbb{R}_+^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (resp. non-negative-definite) real matrices, \( \mathbb{S}^{n \times n} \) denotes the set of \( n \times n \) symmetric real matrices, \( \mathbb{D}^{n \times n} \) denotes the set of \( n \times n \) real matrices with diagonal scalar entries, \((\cdot)^T\) denotes transpose, \((\cdot)^{-1}\) denotes inverse, \(\text{tr}(\cdot)\) denotes the trace operator, \(\|\cdot\|_2\) denotes the Euclidian norm, \(\|\cdot\|_F\) denotes the Frobenius matrix norm, \(\hat{=}\) denotes equality by definition, and \([A]_{ij}\) denotes the \( ij \)-th entry of the matrix \( A \in \mathbb{R}^{n \times m} \). In addition, we write \( \lambda_{\min}(B) \) (resp., \( \lambda_{\max}(B) \)) for the minimum (resp. maximum) eigenvalue of the Hermitian matrix \( B \).

Next, we introduce some fundamental results that are needed to develop the main results of this paper. We begin with the following definition.

**Definition 1.** For a convex hypercube in \( \mathbb{R}^n \) define by \( \Omega = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} \leq \theta_i \leq \theta_i^{\max})_{i = 1, 2, \ldots, n} \} \) where \( (\theta_i^{\min}, \theta_i^{\max}) \) represent the minimum and maximum bounds for the \( i \)-th component of the \( n \)-dimensional parameter vector \( \theta \). Additionally, for a sufficiently small positive constant \( \epsilon \), a second hypercube is defined by \( \Omega_\epsilon = \{ \theta \in \mathbb{R}^n : (\theta_i^{\min} + \epsilon \leq \theta_i \leq \theta_i^{\max} - \epsilon)_{i = 1, 2, \ldots, n} \} \) where \( \Omega_\epsilon \subset \Omega \). Then, the projection operator \( \text{Proj} : \)
$\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is defined component-wise by
\[
\text{Proj}(\theta, y) \triangleq \begin{cases} 
\left( \frac{\theta_i^{\max} - \theta_i}{\epsilon} \right) y_i, & \text{if } \theta_i > \theta_i^{\max} - \epsilon \text{ and } y_i > 0 \\
\left( \frac{\theta_i^{\min} - \theta_i}{\epsilon} \right) y_i, & \text{if } \theta_i < \theta_i^{\min} + \epsilon \text{ and } y_i < 0 \\
y_i, & \text{otherwise}
\end{cases}
\] (1)

where $y \in \mathbb{R}^n$ [13].

It follows from Definition 1 that
\[
(\theta - \theta^*)^T(\text{Proj}(\theta, y) - y) \leq 0, \quad \theta^* \in \mathbb{R}^n,
\] (2)
holds [13,14].

**Remark 1.** Throughout the paper, we use the generalization of this definition to matrices as
\[
\text{Proj}_m(\Theta, Y) = (\text{Proj}(\text{col}_1(\Theta), \text{col}_1(Y)), \ldots, \text{Proj}(\text{col}_m(\Theta), \text{col}_m(Y))),
\] (3)
where $\Theta \in \mathbb{R}^{n \times m}$, $Y \in \mathbb{R}^{n \times m}$, and $\text{col}_i(\cdot)$ denotes the $i$-th column operator. In this case, for a given $\Theta^* \in \mathbb{R}^{n \times m}$, it follows from (2) that
\[
\text{tr} \left[ (\Theta - \Theta^*)^T(\text{Proj}_m(\Theta, Y) - Y) \right] = \sum_{i=1}^m [\text{col}_i(\Theta - \Theta^*)^T(\text{Proj}(\text{col}_i(\Theta), \text{col}_i(Y)) - \text{col}_i(Y))] \leq 0
\] (4)
holds.

We now briefly state the standard model reference control problem. Specifically, consider the uncertain dynamical system given by
\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0,
\] (5)
where $x(t) \in \mathbb{R}^n$ is the state vector available for feedback, $u(t) \in \mathbb{R}^m$ is the control input restricted to the class of admissible controls consisting of measurable functions, $A \in \mathbb{R}^{n \times n}$ is an unknown system matrix, $B \in \mathbb{R}^{n \times m}$ is an unknown input matrix, and the pair $(A, B)$ is controllable.

**Assumption 1.** The unknown control input matrix is parameterized as
\[
B = DA,
\] (6)
where $D \in \mathbb{R}^{n \times m}$ is a known input matrix and $\Lambda \in \mathbb{R}^{m \times m} \cap \mathbb{D}^{m \times m}$ is an unknown control effectiveness matrix which can be decomposed as
\[
\Lambda = I + \delta \Lambda,
\] (7)
where $\delta \Lambda < I$ is unknown.

Next, consider the reference system capturing a desired, ideal closed-loop dynamical system performance
The objective of the model reference adaptive control problem is to construct an adaptive feedback control law given by

$$\dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \quad (8)$$

where $x_r(t) \in \mathbb{R}^n$ is the reference state vector, $c(t) \in \mathbb{R}^m$ is a given uniformly continuous bounded command, $A_r \in \mathbb{R}^{n \times n}$ is the Hurwitz reference system matrix, and $B_r \in \mathbb{R}^{n \times m}$ is the command input matrix. The objective of the model reference adaptive control problem is to construct an adaptive feedback control law $u(t)$ such that the state vector $x(t)$ asymptotically follows the reference state vector $x_r(t)$. We now make the following assumption standard in the model reference adaptive control literature [13,15,16].

**Assumption 2.** There exist gain matrices $K_1 \in \mathbb{R}^{m \times n}$ and $K_2 \in \mathbb{R}^{m \times m}$ such that $A_r = A + DK_1$ and $B_r = DK_2$ hold.

Now, (5) subject to standard Assumptions 1 and 2 yields

$$\dot{x}(t) = A_r x(t) + B_r c(t) + D(I + \delta \Lambda(t))u(t) + DW_1^T x(t) + DW_2^T c(t),$$

where $W_1 \triangleq -K_1^T \in \mathbb{R}^{n \times m}$, $W_2 \triangleq -K_2^T \in \mathbb{R}^{m \times m}$, and $\delta \Lambda \in \mathbb{R}^{m \times m}$ are unknown. Let the adaptive feedback control law be given by

$$u(t) = -(I + \delta \hat{\Lambda}(t))^{-1}(\hat{W}_1^T(t)x(t) + \hat{W}_2^T c(t)), \quad (10)$$

where $\hat{W}_1(t) \in \mathbb{R}^{n \times m}$, $\hat{W}_2(t) \in \mathbb{R}^{m \times m}$, and $\delta \hat{\Lambda}(t) \in \mathbb{R}^{m \times m}$ are the estimates of $W_1$, $W_2$, and $\delta \Lambda$ that respectively satisfy the weight update laws

$$\dot{\hat{W}}_1(t) = \gamma_1 \text{Proj}_m[\hat{W}_1(t), x(t)e^T(t)PD], \quad \hat{W}_1(0) = \hat{W}_{10}, \quad (11)$$

$$\dot{\hat{W}}_2(t) = \gamma_2 \text{Proj}_m[\hat{W}_2(t), c(t)e^T(t)PD], \quad \hat{W}_2(0) = \hat{W}_{20}, \quad (12)$$

$$\delta \dot{\hat{\Lambda}}(t) = \gamma_\Lambda \text{Proj}_m[\delta \hat{\Lambda}(t), D^T Pe(t)u^T(t)], \quad \delta \hat{\Lambda}(0) = \delta \hat{\Lambda}_0, \quad (13)$$

where $\gamma_1 \in \mathbb{R}_+$, $\gamma_2 \in \mathbb{R}_+$, and $\gamma_\Lambda \in \mathbb{R}_+$ are the learning rate gains, $c(t) \triangleq x(t) - x_r(t)$ is the system error state vector, and $P \in \mathbb{R}^{n \times n}$ is a solution of the Lyapunov equation

$$0 = A_r^T P + PA_r + R, \quad (14)$$

with $R \in \mathbb{R}^{n \times n} \cap \mathbb{S}^{n \times n}$. Note that since $A_r$ is Hurwitz, it follows from the converse Lyapunov theory [17] that there exists a unique $P$ satisfying (14) for a given $R$. In addition, the projection bounds are defined such that

$$[\hat{W}_1(t)]_{ij} \leq \hat{W}_{1,\max,i+(j-1)n}, \quad i = 1, ..., n \text{ and } j = 1, ..., m, \quad (15)$$

$$[\hat{W}_2(t)]_{ij} \leq \hat{W}_{2,\max,i+(j-1)m}, \quad i = 1, ..., m \text{ and } j = 1, ..., m, \quad (16)$$

$$[\delta \hat{\Lambda}(t)]_{ij} \leq \delta \hat{\Lambda}_{\max,i+(j-1)m}, \quad i = 1, ..., m \text{ and } j = 1, ..., m \quad (17)$$
where $\hat{W}_{1,\text{max},i+(j-1)m} \in \mathbb{R}^+$, $\hat{W}_{2,\text{max},i+(j-1)m} \in \mathbb{R}^+$, $\delta \hat{\Lambda}_{\text{max},i+(j-1)m} \in \mathbb{R}^+$ denote element-wise projection bounds.

**Remark 2.** The projection bounds on $\delta \hat{\Lambda}(t)$ are selected such that $I + \delta \hat{\Lambda}(t)$ is invertible and therefore (10) is implementable.

Noting that (10) can be given by the equivalent form

$$u(t) = -\hat{W}_1^T(t)x(t) - \hat{W}_2^T(t)c(t) - \delta \hat{\Lambda}(t)u(t), \quad \text{(18)}$$

then (18) can be used in (9) to yield

$$\dot{x}(t) = A_x x(t) + B_x c(t) - D [\hat{W}_1^T(t)x(t) + \hat{W}_2^T(t)c(t) + \delta \hat{\Lambda}(t)u(t)], \quad \text{(19)}$$

and the system error dynamics is then given using (8) and (19) as

$$\dot{e}(t) = A_e e(t) - D [\hat{W}_1^T(t)x(t) + \hat{W}_2^T(t)c(t) + \delta \hat{\Lambda}(t)u(t)], \quad e(0) = e_0, \quad \text{(20)}$$

where $\hat{W}_1(t) \triangleq \hat{W}_1(t) - W_1 \in \mathbb{R}^{n \times m}$, $\hat{W}_2(t) \triangleq \hat{W}_2(t) - W_2 \in \mathbb{R}^{m \times m}$, and $\delta \hat{\Lambda}(t) \triangleq \delta \hat{\Lambda}(t) - \delta \Lambda \in \mathbb{R}^{m \times m}$.

**Remark 3.** The weight update laws given by (11), (12), and (13) can be derived using Lyapunov analysis by considering the Lyapunov function candidate given by (see, for example, [13, 15, 16])

$$V(e, \hat{W}_1, \hat{W}_2, \delta \hat{\Lambda}) = e^T P e + \gamma_1^{-1} \text{tr} \hat{W}_1^T \hat{W}_1 + \gamma_2^{-1} \text{tr} \hat{W}_2^T \hat{W}_2 + \gamma_3^{-1} \text{tr} \delta \hat{\Lambda}^T \delta \hat{\Lambda}. \quad \text{(21)}$$

Note that $V(0, 0, 0, 0) = 0$ and $V(e, \hat{W}_1, \hat{W}_2, \delta \hat{\Lambda}) > 0$ for all $(e, \hat{W}_1, \hat{W}_2, \delta \hat{\Lambda}) \neq (0, 0, 0, 0)$. Now, differentiating (21) yields

$$\dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t)) = -e^T(t) \text{Re}(t) - 2e^T(t)PD\hat{W}_1^T(t)x(t) - 2e^T(t)PD\hat{W}_2^T(t)c(t) - 2e^T(t)PD\delta \hat{\Lambda}(t)u(t) \quad \text{(22)}$$

where using (11), (12), and (13) in (22) results in $\dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t)) \leq -e^T(t) \text{Re}(t) \leq 0$, which guarantees that the system error state vector $e(t)$ and the weight errors $\hat{W}_1(t), \hat{W}_2(t)$, and $\delta \hat{\Lambda}(t)$ are Lyapunov stable, and are therefore bounded for all $t \in \mathbb{R}^+$. Since $x(t)$ and $c(t)$ are bounded for all $t \in \mathbb{R}^+$, it follows from (20) that $e(t)$ is bounded, and hence, $\dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t))$ is bounded for all $t \in \mathbb{R}^+$. It then follows from Barbalat’s lemma that $\lim_{t \to \infty} \dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t)) = 0$, which consequently shows that $e(t) \to 0$ as $t \to \infty$. 

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III. Actuator Dynamics with Uncertain Control Effectiveness

For the model reference adaptive control framework introduced in the previous section, we now introduce the actuator dynamics problem. Consider the uncertain system given by

\[ \dot{x}(t) = Ax(t) + Bv(t), \quad x(0) = x_0, \quad \text{(23)} \]

where \( v(t) \in \mathbb{R}^m \) is the actuator output given by

\[ \dot{v}(t) = M(u(t) - v(t)), \quad v(0) = v_0, \quad \text{(24)} \]

where \( M \in \mathbb{R}^{m \times m} \cap \mathbb{S}^{m \times m} \) with diagonal entries \( \lambda_{i,i} > 0, \ i = 1, \cdots, m \), which represent the actuator bandwidth of each control channel. Using Assumptions 1 and 2, (23) can be equivalently written as

\[ \dot{x}(t) = A_r x(t) + B_r c(t) + D [(I + \delta \Lambda) u(t) + W_1^T(t)x(t) + W_2^T(t)c(t)] + D (I + \delta \Lambda) [v(t) - u(t)] \]

\[ = A_r x(t) + B_r c(t) + D [u(t) + W_3^T(t)x(t) + W_4^T(t)c(t) + \delta \Lambda v(t)] + D [v(t) - u(t)], \quad \text{(25)} \]

where \( W_1 \triangleq -K_1 \in \mathbb{R}^{n \times m}, W_2 \triangleq -K_2 \in \mathbb{R}^{m \times m}, \) and \( \delta \Lambda \in \mathbb{R}^{m \times m} \) are unknown. Now, let the adaptive feedback control law be given by

\[ u(t) = -\dot{W}_1^T(t)x(t) - \dot{W}_2^T(t)c(t) - \dot{\delta \Lambda}(t)v(t), \quad \text{(26)} \]

where \( \dot{W}_1(t) \in \mathbb{R}^{n \times m}, \dot{W}_2(t) \in \mathbb{R}^{m \times m}, \) and \( \dot{\delta \Lambda}(t) \in \mathbb{R}^{m \times m} \) satisfy the respective weight update laws

\[ \dot{\bar{W}}_1(t) = \gamma_1 \text{Proj}_m [\bar{W}_1(t), \ x(t)e^T(t)PD], \quad \bar{W}_1(0) = \bar{W}_{10}, \quad \text{(27)} \]

\[ \dot{\bar{W}}_2(t) = \gamma_2 \text{Proj}_m [\bar{W}_2(t), \ c(t)e^T(t)PD], \quad \bar{W}_2(0) = \bar{W}_{20}, \quad \text{(28)} \]

\[ \dot{\delta \Lambda}(t) = \gamma_\Lambda \text{Proj}_m [\delta \Lambda(t), \ D^T Pe(t)v^T(t)], \quad \delta \Lambda(0) = \delta \Lambda_0, \quad \text{(29)} \]

with the projection bounds defined respectively by (15), (16), and (17), where the projection bounds of \( \delta \Lambda(t) \) are chosen such that

\[ M \delta \Lambda^T(t) + \delta \Lambda(t)M > -2M \quad \text{(30)} \]

holds. It follows that using (26) in (25), the system dynamics are now given by

\[ \dot{x}(t) = A_r x(t) + B_r c(t) - D [\dot{W}_1^T(t)x(t) + \dot{W}_2^T(t)c(t) + \delta \Lambda(t)v(t)] + D [v(t) - u(t)]. \quad \text{(31)} \]

**Remark 4.** Note that to show the condition given by (30) holds, we consider \( \delta \Lambda_{i_1, \cdots, i_r} \in \mathbb{R}^{m \times m} \) defined
Consider the following modified reference system,

\[ A \text{ Known Actuator Output Case} \]

and using (31) and (34) the system error dynamics are given by

\[
\dot{e}(t) = A_r e(t) - D [\hat{W}_1^T(t)x(t) + \hat{W}_2^T(t)c(t) + \delta \hat{A}(t)v(t)] + \delta \hat{A}(t) + \hat{A}_r(0) = x_{r0},
\]

where \( r \in \{1, 2\} \), such that \( r \in \{1, ..., mm\} \) and \( \delta \hat{A}_{r_1,...,r_r} \) represents the corners of the hypercube defining the maximum variation of \( \delta \hat{A}(t) \). It then follows that if (32) satisfies the inequality

\[
M \delta \hat{A}_{r_1,...,r_r}^T + \delta \hat{A}_{r_1,...,r_r} > -2M
\]

for all permutations of \( \delta \hat{A}_{r_1,...,r_r} \), then (30) holds.

In what follows, we first consider the case in which the actuator output is known (Section A) and then generalize our results to the case where it is unknown (Section B).

### A. Known Actuator Output Case

Consider the following modified reference system,

\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t) + D [v(t) - u(t)], \quad x_r(0) = x_{r0}, \quad (34)
\]

and using (31) and (34) the system error dynamics are given by

\[
\dot{e}(t) = A_r e(t) - D [\hat{W}_1^T(t)x(t) + \hat{W}_2^T(t)c(t) + \delta \hat{A}(t)v(t)] + \delta \hat{A}(t) + \hat{A}_r(0) = x_{r0}, \quad (35)
\]

**Theorem 1.** Consider the uncertain dynamical system given by (23) subject to Assumptions 1 and 2, the reference system given by (34), the actuator dynamics given by (24), the adaptive feedback control law given by (26) along with the update laws (27), (28), and (29). In addition, let

\[
A(\hat{W}_1(t), \delta \hat{A}(t), M) = \begin{bmatrix} A_r + D \hat{W}_1^T(t) & D(I + \delta \hat{A}(t))M \\ -\hat{W}_1^T(t) & -(I + \delta \hat{A}(t))M \end{bmatrix}
\]

be quadratically stable. Then, the solution \((e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{A}(t), x_r(t), v(t))\) of the closed-loop dynamical system are bounded and \(\lim_{t \to \infty} e(t) = 0\).

**Proof.** To show Lyapunov stability and guarantee boundedness of the system error state \(e(t)\), the weight errors \(\hat{W}_1(t)\) and \(\hat{W}_2(t)\), and the control effectiveness error \(\delta \hat{A}(t)\), consider the Lyapunov function candidate given by

\[
V(e, \hat{W}_1, \hat{W}_2, \delta \hat{A}) = e^T Pe + \gamma_1^{-1} \text{tr} \hat{W}_1^T \hat{W}_1 + \gamma_2^{-1} \text{tr} \hat{W}_2^T \hat{W}_2 + \gamma_\hat{A}^{-1} \text{tr} \delta \hat{A}^T \delta \hat{A}. 
\]
Note that \( \mathcal{V}(0,0,0,0) = 0 \) and \( \mathcal{V}(e, \dot{W}_1, \dot{W}_2, \dot{\Lambda}) > 0 \) for all \((e, \dot{W}_1, \dot{W}_2, \dot{\Lambda}) \neq (0,0,0,0)\). Differentiating (37) yields \( \dot{\mathcal{V}}(e(t), \dot{W}_1(t), \dot{W}_2(t), \dot{\Lambda}(t)) \leq -e^T(t)Re(t) \leq 0 \), which guarantees the Lyapunov stability, and hence, the boundedness of the solution \((e(t), \dot{W}_1(t), \dot{W}_2(t), \dot{\Lambda}(t))\).

To show the boundedness of \(x(t)\) and \(v(t)\), first note that the actuator dynamics (24) can be transformed into the equivalent form

\[
\dot{v}(t) = -M\ddot{v}(t) + u(t).
\]

where \(\ddot{v}(t) = M^{-1}v(t)\). Using (38), the reference system (34) subject to (26) can be written as

\[
\dot{x}_r(t) = A_r x_r(t) + B_r c(t) + D [M \ddot{v}(t) + \dot{W}_1^T(t)x(t) + \dot{W}_2^T(t)c(t) + \dot{\Lambda}(t)M\ddot{v}(t)],
\]

\[
= A_r x_r(t) + B_r c(t) + D [M \ddot{v}(t) + \dot{W}_1^T(t)\dot{e}(t) + \dot{W}_1^T(t)x_r(t)
+ \dot{W}_2^T c(t) + \dot{\Lambda}(t)M\ddot{v}(t)]
\]

and the actuator dynamics subject to (26) follow as

\[
\dot{\ddot{v}}(t) = -M\ddot{v}(t) - \dot{W}_1^T(t)x(t) - \dot{W}_2^T c(t) - \dot{\Lambda}(t)M\ddot{v}(t),
\]

\[
= -M\ddot{v}(t) - \dot{W}_1^T(t)\dot{e}(t) - \dot{W}_1^T(t)x_r(t) - \dot{W}_2^T c(t) - \dot{\Lambda}(t)M\ddot{v}(t).
\]

Then, (39) and (40) can be rewritten in compact form as

\[
\dot{\xi}(t) = A(\dot{W}_1(t), \dot{\Lambda}(t), M)\xi(t) + \omega(\cdot),
\]

with \(\xi(t) = [x_r(t), \ddot{v}(t)]^T\) and

\[
\omega(\cdot) = \begin{bmatrix}
B_r c(t) + D\dot{W}_1^T(t)e(t) + D\dot{W}_2^T(t)c(t) \\
-\dot{W}_1^T(t)e(t) - \dot{W}_2^T(t)c(t)
\end{bmatrix}.
\]

Note that \(\omega(\cdot)\) in (41) is a bounded perturbation as a result of Lyapunov stability of the quadruple \((e(t), \dot{W}_1(t), \dot{W}_2(t), \dot{\Lambda}(t))\). Now, it follows that since \(\omega(\cdot)\) is bounded and \(A(\dot{W}_1(t), \dot{\Lambda}(t), M)\) is quadratically stable, then \(x_r(t)\) and \(\ddot{v}(t)\) are also bounded (see, for example, [18]), additionally it follows that \(\ddot{v}(t)\) is bounded as a result of the boundedness of \(\dddot{v}(t)\).

Finally, since \(e(t)\) and \(x_r(t)\) are bounded, this implies \(x(t)\) is bounded. In addition, it follows from (35) that \(\dot{e}(t)\) is bounded, and hence, \(\dot{\mathcal{V}}(e(t), \dot{W}_1(t), \dot{W}_2(t), \dot{\Lambda}(t))\) is bounded. As a consequence of the boundedness of \(\dot{\mathcal{V}}(e(t), \dot{W}_1(t), \dot{W}_2(t), \dot{\Lambda}(t))\) and Barbalat’s lemma [18], \(\lim_{t \to \infty} e(t) = 0\) is immediate.  

\textbf{Remark 5.} The previous proof uses a transformation of the actuator dynamics, stating that the two forms are equivalent. To show this consider the actuator dynamics given by (38) and let \(\dddot{v}(t) = M^{-1}v(t)\)
which implies \( v(t) = M\tilde{v}(t) \). It then follows that

\[
\dot{v}(t) = M\dot{\tilde{v}}(t),
\]

\[
= M(-M\tilde{v}(t) + u(t)),
\]

\[
= M(-MM^{-1}v(t) + u(t)),
\]

\[
= M(-v(t) + u(t)),
\]

which results in the actuator dynamics given by (24).

**Remark 6.** It is important to note that the transformation of the actuator dynamics is used such that the resulting form of (36) simplifies the LMI conditions used in later analysis.

**Remark 7.** For the results given in Theorem 1 to hold, it is assumed that (36) is quadratically stable [19]. To show the feasibility of this condition, consider the Lyapunov inequality given by

\[
A^T(\hat{W}_1(t), \delta\hat{\Lambda}(t), M)P + PA(\hat{W}_1(t), \delta\hat{\Lambda}(t), M) < 0,
\]

\[
P = P^T > 0
\]

with

\[
P = \begin{bmatrix}
P & PD \\
D^TP & D^TPD + \rho I
\end{bmatrix},
\]

where \( P \in R^{n \times n}_+ \cap S^{n \times n} \) is a solution of the Lyapunov equation given by (14) with \( R \in R^{n \times n}_+ \cap S^{n \times n} \) and \( \rho \in R_+ \). Note that the positive-definiteness of (44) follows from the positive-definiteness of \( P \) and the positive-definiteness of the Schur complement of (44) given by

\[
S_1 = D^TPD + \rho I - D^TP(P)^{-1}PD = \rho I > 0.
\]

Next, consider

\[
Q = A^T(\hat{W}_1(t), \delta\hat{\Lambda}(t), M)P + PA(\hat{W}_1(t), \delta\hat{\Lambda}(t), M)
\]

\[
= \begin{bmatrix}
-R & A_t^TPD - \rho\hat{W}_1(t) \\
D^TPA_t - \rho\hat{W}_1^{T}(t) & -2\rho M - \rho(M\delta\hat{\Lambda}(t) + \delta\hat{\Lambda}(t)M)
\end{bmatrix}.
\]

Noting that \(-R\) is negative definite, we then consider the Schur complement of (46) given as

\[
S_2 = \left[-\rho(2M + M\delta\hat{\Lambda}(t) + \delta\hat{\Lambda}(t)M)\right] + [D^TPA_t - \rho\hat{W}_1^{T}(t)]R^{-1}\left[A_t^TPD - \rho\hat{W}_1(t)\right],
\]

where using the condition on the projection bounds of \( \delta\hat{\Lambda}(t) \) given by (30) it is guaranteed that (47) is a negative-definite matrix for large values of \( M \). Therefore, quadratic stability of (36) holds for large values of \( M \).
Remark 8. We now utilize LMIs to satisfy the quadratic stability of (36) for given projection bounds $\hat{W}_{1,\max}$ and $\delta \hat{\Lambda}_{\max}$ for the elements of $\hat{W}_1(t)$ and $\delta \hat{\Lambda}(t)$, respectively, and the bandwidths of the actuator dynamics $M$. For this purpose, let $\delta \hat{\Lambda}_{1,\ldots,i_r} \in \mathbb{R}^{m \times m}$ be given by (32) and $\hat{W}_{1,\ldots,i_l} \in \mathbb{R}^{n \times m}$ be defined as

$$\hat{W}_{1,\ldots,i_l} = \begin{bmatrix}
(-1)^{i_1} \hat{W}_{1,\max,1} & (-1)^{i_2} \hat{W}_{1,\max,1+n} & \ldots & (-1)^{i_{l+n-1}} \hat{W}_{1,\max,1+(m-1)n} \\
(-1)^{i_2} \hat{W}_{1,\max,2} & (-1)^{i_3} \hat{W}_{1,\max,2+n} & \ldots & (-1)^{i_{l+n-1}} \hat{W}_{1,\max,2+(m-1)n} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{i_n} \hat{W}_{1,\max,n} & (-1)^{i_1} \hat{W}_{1,\max,2n} & \ldots & (-1)^{i_{l+(m-1)n}} \hat{W}_{1,\max,mn}
\end{bmatrix}$$

(48)

where $i_l \in \{1, 2\}, l \in \{1, \ldots, mn\}$ such that $\hat{W}_{1,\ldots,i_l}$ and $\delta \hat{\Lambda}_{1,\ldots,i_r}$ represent the corners of the hypercube defining the maximum variation of $\hat{W}_1(t)$ and $\delta \hat{\Lambda}(t)$. Following the results in [4, 6, 19], if

$$A_{i_1,\ldots,i_{l+r}} = \begin{bmatrix}
A_r + D\hat{W}_{1,\ldots,i_l}^T & D(I + \delta \hat{\Lambda}_{1,\ldots,i_r})M \\
-\hat{W}_{1,\ldots,i_l}^T & -(I + \delta \hat{\Lambda}_{1,\ldots,i_r})M
\end{bmatrix},$$

satisfies the matrix inequality

$$A_{i_1,\ldots,i_{l+r}}^T \mathcal{P} + \mathcal{P} A_{i_1,\ldots,i_{l+r}} < 0, \quad \mathcal{P} = \mathcal{P}^T > 0$$

(50)

for all permutations of $\hat{W}_{1,\ldots,i_l}$ and $\delta \hat{\Lambda}_{1,\ldots,i_r}$, then (36) is quadratically stable. We then cast (50) as a convex optimization problem given by

$$\text{minimize } M, \quad \text{subject to } (50).$$

(51)

(52)

It should be noted, this convex optimization problem characterizes the fundamental interplay between the allowable system uncertainties (through the selection of the projection operator bounds) and the bandwidth of the actuator dynamics.

B. Unknown Actuator Output Case

We now extend the results of the previous section to the case of unknown actuator output. For this purpose, first consider the modified adaptive feedback control law given by

$$u(t) = -\hat{W}_{1}^T(t)x(t) - \hat{W}_{2}^T(t)c(t) - \delta \hat{\Lambda}(t)\hat{v}(t),$$

(53)

where $\hat{W}_{1}(t)$ and $\hat{W}_{2}(t)$ satisfy the weight update laws given by (27) and (28) respectively and for the case of unknown actuator output, the update law given by (29) is modified as

$$\dot{\delta} \hat{\Lambda}(t) = \gamma \Lambda \text{Proj}_{\mathbb{R}^m}[\delta \hat{\Lambda}(t), D^T P e(t)\hat{v}(t)], \quad \delta \hat{\Lambda}(0) = \delta \hat{\Lambda}_0,$$

(54)
with the projection bounds defined by (17). Additionally, \( \hat{v} \in \mathbb{R}^m \) is an estimate of the unknown actuator output \( v(t) \), satisfying the update law,

\[
\dot{\hat{v}}(t) = \mu(I + \delta \hat{\Lambda}(t))D^TPe(t) + M(u(t) - \hat{v}(t)), \quad \hat{v}(0) = \hat{v}_0,
\]

with \( \mu = \beta M, \beta \in \mathbb{R}_+ \) being a design parameter. Next, consider the modified reference system given by

\[
x_r(t) = A_r x_r(t) + B_r c(t) + D[\hat{v}(t) - u(t)], \quad x_r(0) = x_{r0},
\]

Now, using the system dynamics (25) with the adaptive feedback control law (53) and the reference system (56), the system error dynamics are given by

\[
\begin{aligned}
\dot{e}(t) &= A_r e(t) - D[\hat{W}_1^T(t)x(t) + \hat{W}_2^T(t)c(t)] + D\delta \Lambda e(t) - D\delta \hat{\Lambda}(t)e(t) - D\hat{v}(t), \quad e(0) = e_0,
\end{aligned}
\]

where \( \hat{v}(t) \triangleq \hat{v}(t) - v(t) \in \mathbb{R}^m \).

**Theorem 2.** Consider the uncertain dynamical system given by (23) subject to Assumptions 1 and 2, the reference system given by (56), the actuator dynamics given by (24), the adaptive feedback control law given by (53) along with the update laws (27), (28), (54), and (55). In addition, let

\[
\begin{aligned}
\mathcal{A}(\hat{W}_1(t), \delta \hat{\Lambda}(t), M) &= \begin{bmatrix} A_r + D\hat{W}_1^T(t) & D(I + \delta \hat{\Lambda}(t))M \\ -\hat{W}_1^T(t) & -(I + \delta \hat{\Lambda}(t))M \end{bmatrix},
\end{aligned}
\]

be quadratically stable. If

\[
\frac{\beta}{\lambda_{\min}(R)} \|PD\|^2_F w_\Lambda^2 < 1
\]

holds, where \( \|\delta \hat{\Lambda}(t)\|_F \leq w_\Lambda^* \) denotes an upper bound, then the solution \( (e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t), x_r(t), v(t), \hat{v}(t)) \) of the closed-loop dynamical system are bounded and \( \lim_{t \to \infty} e(t) = 0 \).

**Proof.** To show Lyapunov stability and guarantee boundedness of the system error state \( e(t) \), the weight uncertainties \( \hat{W}_1(t) \) and \( \hat{W}_2(t) \), the control effectiveness error \( \delta \hat{\Lambda}(t) \), and the actuator output error \( \hat{v}(t) \), consider the Lyapunov function candidate given by

\[
\begin{aligned}
\mathcal{V}(e, \hat{W}_1, \hat{W}_2, \delta \hat{\Lambda}, \hat{v}) &= e^TPe + \gamma_1^{-1} \text{tr} \, \hat{W}_1^T \hat{W}_1 + \gamma_2^{-1} \text{tr} \, \hat{W}_2^T \hat{W}_2 \\
&\quad + \gamma_{\Lambda}^{-1} \text{tr} \, \delta \hat{\Lambda}^T \delta \hat{\Lambda} + \beta^{-1} \hat{v}^TM^{-1}\hat{v}.
\end{aligned}
\]

Note that \( \mathcal{V}(0, 0, 0, 0, 0) = 0 \) and \( \mathcal{V}(e, \hat{W}_1, \hat{W}_2, \delta \hat{\Lambda}, \hat{v}) > 0 \) for all \( (e, \hat{W}_1, \hat{W}_2, \delta \hat{\Lambda}, \hat{v}) \neq (0, 0, 0, 0, 0) \). Differenti-
ating (60) yields

\[
\dot{V}(e(t), \dot{W}_1(t), \dot{W}_2(t), \delta \hat{\Lambda}(t), \ddot{v}(t))
\]

\[
= -e^T(t)Re(t) - 2e^T(t)PD\dot{\hat{\Lambda}}^T(t)x(t) - 2e^T(t)PD\dot{\hat{\Lambda}}^T(t)c(t) + 2e^T(t)PD\delta \Lambda v(t)
\]

\[
-2e^T(t)PD\delta \Lambda(t)\dot{v}(t) - 2e^T(t)PD\dot{v}(t) + 2\gamma_1^{-1}\text{tr} \ W_1^T(t)\dot{W}_1(t) + 2\gamma_2^{-1}\text{tr} \ W_2^T(t)\dot{W}_2(t)
\]

\[
+2\gamma_{\Lambda}^{-1}\text{tr} \ \delta \hat{\Lambda}^T(t)\delta \hat{\Lambda}(t) + 2\beta^{-1}\ddot{v}(t)M^{-1}\ddot{v}(t),
\]

(61)

where using (27) and (28), it follows that (61) reduces to

\[
\dot{V}(e(t), \dot{W}_1(t), \dot{W}_2(t), \delta \hat{\Lambda}(t), \ddot{v}(t))
\]

\[
\leq -e^T(t)Re(t) + 2e^T(t)PD\delta \Lambda v(t) - 2e^T(t)PD\delta \Lambda(t)\dot{v}(t) - 2e^T(t)PD\dot{v}(t)
\]

\[
+2\gamma_{\Lambda}^{-1}\text{tr} \ \delta \hat{\Lambda}^T(t)\delta \hat{\Lambda}(t) + 2\beta^{-1}\ddot{v}(t)M^{-1}\ddot{v}(t).
\]

(62)

This can equivalently be expressed as

\[
\dot{V}(e(t), \dot{W}_1(t), \dot{W}_2(t), \delta \hat{\Lambda}(t), \ddot{v}(t))
\]

\[
\leq -e^T(t)Re(t) + 2e^T(t)PD\delta \Lambda v(t) - 2e^T(t)PD\delta \Lambda(t)\dot{v}(t) + 2e^T(t)PD\dot{v}(t)
\]

\[
-2e^T(t)PD\delta \Lambda\dot{v}(t) - 2e^T(t)PD\dot{v}(t) + 2e^T(t)PD\delta \Lambda(t)\dot{v}(t) - 2e^T(t)PD\delta \Lambda(t)\ddot{v}(t)
\]

\[
+2\gamma_{\Lambda}^{-1}\text{tr} \ \delta \hat{\Lambda}^T(t)\delta \hat{\Lambda}(t) + 2\beta^{-1}\ddot{v}(t)M^{-1}\ddot{v}(t),
\]

= -e^T(t)Re(t) - 2e^T(t)PD\delta \Lambda\dot{v}(t) + 2\gamma_{\Lambda}^{-1}\text{tr} \ \delta \hat{\Lambda}^T(t)\delta \hat{\Lambda}(t) + 2e^T(t)PD\delta \Lambda\ddot{v}(t)

\[
+2e^T(t)PD(I + \delta \Lambda(t))\ddot{v}(t) + 2\beta^{-1}\ddot{v}(t)M^{-1}\ddot{v}(t)
\]

(63)

Finally, noting that \(\dot{\hat{v}}(t) = \hat{v}(t) - \ddot{v}(t)\), and using the actuator dynamics given by (24) along with the update laws (54) and (55) yields

\[
\dot{V}(e(t), \dot{W}_1(t), \dot{W}_2(t), \delta \hat{\Lambda}(t), \ddot{v}(t))
\]

\[
\leq -e^T(t)Re(t) + 2e^T(t)PD\delta \hat{\Lambda}(t)\ddot{v}(t) - 2\beta^{-1}\ddot{v}(t)M^{-1}\ddot{v}(t),
\]

\[
\leq -\lambda_{\text{min}}( \|e(t)\|^2_2 + 2\|e(t)\|_2\|PD\|_F\|\delta \hat{\Lambda}(t)\|_F\|\ddot{v}(t)\|_2 - 2\beta^{-1}\|\ddot{v}(t)\|_2
\]

\[
\leq -\lambda_{\text{min}}(R)(\|e(t)\|^2_2 + 2\|e(t)\|_2\|PD\|_F \ w_\Lambda^*\|\ddot{v}(t)\|_2 - 2\beta^{-1}\|\ddot{v}(t)\|_2
\]

(64)

where \(\|\delta \hat{\Lambda}(t)\|_F \leq w_\Lambda^*\) holds due to projection operator. Now, using Young's inequality [20] on the second term yields

\[
\dot{V}(e(t), \dot{W}_1(t), \dot{W}_2(t), \delta \hat{\Lambda}(t), \ddot{v}(t))
\]

\[
\leq -\lambda_{\text{min}}(R)(\|e(t)\|^2_2 + 2\|e(t)\|_2 + \frac{1}{\alpha}\|PD\|_F^2 \ w_\Lambda^{2*}\|\ddot{v}(t)\|_2^2 - 2\beta^{-1}\|\ddot{v}(t)\|_2^2.
\]

(65)
Let $\alpha = \frac{1}{2}\lambda_{\min}(R)$, it follows

$$
\dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t), \hat{v}(t)) \\
\leq -\frac{\lambda_{\min}(R)}{2} \|e(t)\|_2^2 - 2\beta^{-1}\left[1 - \frac{\beta}{\lambda_{\min}(R)}\right] \|PD\|_w^2 \|\hat{v}(t)\|_2^2.
$$

(66)

Using condition (59) in (66), it follows that $\dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t), \hat{v}(t)) \leq 0$, which guarantees the Lyapunov stability, and hence, the boundedness of the solution $(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t), \hat{v}(t))$.

To show the boundedness of $x_r(t)$ and $v(t)$, using the transformed actuator dynamics (38) consider the reference system (56) subject to (53) as

$$
x_r(t) = A_r x_r(t) + B_r c(t) + D \left[ \hat{v}(t) + \delta \hat{\Lambda}(t) \hat{v}(t) + \hat{W}_1^T(t) x(t) + \hat{W}_2^T c(t) \right],
$$

$$
= A_r x_r(t) + B_r c(t) + D \left[ \hat{v}(t) + \delta \hat{\Lambda}(t) \hat{v}(t) + \hat{W}_1^T(t) c(t) + \hat{W}_1^T(t) x_r(t) + \hat{W}_2^T c(t) + v(t) - v(t) + \delta \hat{\Lambda}(t) v(t) - \delta \hat{\Lambda}(t) v(t) \right],
$$

where using $v(t) = M \hat{v}(t)$ and $\hat{v}(t) = \hat{v}(t) - v(t)$, it follows from collecting terms that

$$
x_r(t) = A_r x_r(t) + B_r c(t) + D \left[ M \hat{v}(t) + \delta \hat{\Lambda}(t) M \hat{v}(t) + \hat{W}_1^T(t) c(t) + \hat{W}_1^T(t) x_r(t) + \hat{W}_2^T c(t) + \delta \hat{\Lambda}(t) \hat{v}(t) + D \hat{v}(t).\right.
$$

(67)

The transformed actuator dynamics given by (38) subject to (53) follow as

$$
\dot{v}(t) = -M \hat{v}(t) - \hat{W}_1^T(t) x(t) - \hat{W}_2^T c(t) - \delta \hat{\Lambda}(t) \hat{v}(t),
$$

$$
= -M \hat{v}(t) - \hat{W}_1^T(t) x(t) - \hat{W}_2^T c(t) - \delta \hat{\Lambda}(t) \hat{v}(t) + \delta \hat{\Lambda}(t) v(t) - \delta \hat{\Lambda}(t) v(t)
$$

$$
= -M \hat{v}(t) - \delta \hat{\Lambda}(t) M \hat{v}(t) - \hat{W}_1^T(t) c(t) - \hat{W}_1^T(t) x_r(t) - \hat{W}_2^T c(t) - \delta \hat{\Lambda}(t) \hat{v}(t).
$$

(69)

where (68) and (69) can be rewritten in compact form as

$$
\dot{\xi}(t) = A(\hat{W}_1(t), \delta \hat{\Lambda}(t), M) \xi(t) + \omega(\cdot),
$$

(70)

with $\xi(t) = [x_r(t), \hat{v}(t)]^T$ and

$$
\omega(\cdot) = \left[ B_r c(t) + D \hat{W}_1^T(t) c(t) + D \hat{W}_2^T(t) c(t) + D \delta \hat{\Lambda}(t) \hat{v}(t) + D \hat{v}(t) \right.
$$

$$
- \hat{W}_1^T(t) c(t) - \hat{W}_2^T(t) c(t) - \delta \hat{\Lambda}(t) \hat{v}(t).\right]
$$

(71)

Note that $\omega(\cdot)$ in (70) is a bounded perturbation as a result of Lyapunov stability of the quintuple $(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t), \hat{v}(t))$. Now, it follows that since $\omega(\cdot)$ is bounded and $A(\hat{W}_1(t), \delta \hat{\Lambda}(t), M)$ is quadratically stable, then $x_r(t)$ and $\hat{v}(t)$ are also bounded (see, for example, [18]), additionally it follows that $v(t)$ is bounded as a result of the boundedness of $\hat{v}(t)$.

Finally, since $e(t)$ and $x_r(t)$ are bounded, this implies $x(t)$ is bounded. In addition, it follows from (57) that $\hat{v}(t)$ is bounded, and hence, $\dot{V}(e(t), \hat{W}_1(t), \hat{W}_2(t), \delta \hat{\Lambda}(t), \hat{v}(t))$ is bounded. As a consequence of
the boundedness of $\bar{V}(e(t), \tilde{W}_1(t), \tilde{W}_2(t), \delta \hat{\Lambda}(t), \bar{\delta}(t))$ and Barbalat’s lemma [18], $\lim_{t \to \infty} e(t) = 0$ is now immediate.

Remark 9. As the quadratic stability condition given by (58) is the same as (36) from the previous section, it should be noted that the same process and results presented in Remarks 7 and 8 hold for the case of unknown actuator output.

IV. Illustrative Example

In order to illustrate the proposed adaptive control architecture with actuator dynamics, consider a second-order system

$$
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0.5 & 0.5
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix} \Lambda v(t),
$$

(72)

with $\Lambda = 0.5$ denoting the unknown control effectiveness and we assume that the actuator output is not available. For this example, let $x_1(t)$ represent the angle in radians and $x_2(t)$ represent the angular rate of change in radians per second. We select a reference system with a natural frequency of $\omega_n = 1.0$ rad/s and a damping ratio $\zeta = 0.7$ such that

$$
A_r =
\begin{bmatrix}
0 & 1 \\
-1 & -1.4
\end{bmatrix},
B_r =
\begin{bmatrix}
0 \\
1
\end{bmatrix},
$$

(73)

hold and the projection operator bounds are set to 2.5 for the unknown weights $W_1$ and $W_2$, and 0.5 for the unknown control effectiveness $\delta \hat{\Lambda}$. Furthermore, the tracking command $c(t)$ is applied through a first order filter and we use a single actuator bandwidth for the control input such that $M = \lambda$, $\lambda \in \mathbb{R}_+$. Utilizing the LMI analysis highlighted in Remark 8, the minimum allowable actuator bandwidth is calculated as $\lambda_{\text{min}} = 6.63$. Furthermore, we set $R = I_2$ for both the standard adaptive controller design (see (10), (11), (12), and (13)) and the proposed controller design (we use the results of Theorem 2).

Figures 1–3 show the standard adaptive controller performance in the presence of actuator dynamics using a range of actuator bandwidths. It can be seen from the Figures that due to the introduced actuator dynamics, the standard controller performance becomes worse as the actuator bandwidth is decreased. Figures 4–6 show the proposed controller design with the same actuator bandwidth values used as with the standard controller case. Since it was calculated that the minimum actuator bandwidth allowed for the actuator dynamics was 6.63, it is expected that the system performances are acceptable (and clearly better than the results in Figures 1–3) in the respective figures.

In Figures 7 and 8, we let the actuator bandwidth be smaller than 6.63. In Figure 7, it can be seen that the closed-loop system still remains stable for $\lambda = 3.5$; however, it is unstable in Figure 8 for $\lambda = 3$. This is consistent with the presented theory, as we provide a (conservative) upper bound on the allowable actuator bandwidth such that the closed-loop system remains stable.
Figure 1. MRAC performance with actuator dynamics ($\lambda = 25, \gamma = 10, \gamma_2 = 10, \gamma_\Lambda = 1$).

Figure 2. MRAC performance with actuator dynamics ($\lambda = 10, \gamma = 10, \gamma_2 = 10, \gamma_\Lambda = 1$).
Figure 3. MRAC performance with actuator dynamics ($\lambda = 6.63$, $\gamma = 10$, $\gamma_2 = 10$, $\gamma_{\Lambda} = 1$).

Figure 4. Proposed controller performance with actuator dynamics ($\lambda = 25$, $\gamma_1 = 10$, $\gamma_2 = 10$, $\gamma_{\Lambda} = 1$, $\beta = 0.25$).
Figure 5. Proposed controller performance with actuator dynamics ($\lambda = 10, \gamma_1 = 10, \gamma_2 = 10, \gamma_\Lambda = 1, \beta = 0.25$).

Figure 6. Proposed controller performance with actuator dynamics ($\lambda = 6.63, \gamma_1 = 10, \gamma_2 = 10, \gamma_\Lambda = 1, \beta = 0.25$).
Figure 7. Proposed controller performance with actuator dynamics ($\lambda = 3.5, \gamma_1 = 10, \gamma_2 = 10, \gamma_A = 1, \beta = 0.25$).

Figure 8. Proposed controller performance with actuator dynamics ($\lambda = 3, \gamma_1 = 10, \gamma_2 = 10, \gamma_A = 1, \beta = 0.25$).
V. Conclusion

It is well known that the presence of actuator dynamics can seriously limit the stability and achievable performance of model reference adaptive controllers. In this paper, we generalized an LMI-based hedging approach to consider the case in which control effectiveness is unknown. Specifically, it was shown to maintain stability of adaptive controllers in the presence of actuator dynamics. Using LMIs, it is analytically proved that the closed-loop dynamical system, including the modified reference model trajectory, is stable.

References


