Direct Adaptive Uncertainty Minimization Framework in the Presence of Unknown Control Effectiveness

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In recent work, a direct uncertainty minimization framework has been developed and demonstrated for model reference adaptive controllers, which consists of a new architecture involving modification terms in the adaptive controller and the update law. Specifically, these modification terms are constructed through a gradient minimization procedure in order to achieve improved closed-loop system performance with adaptive controllers. In this paper, we generalize this framework for dynamical systems subject to uncertainty in the control effectiveness and provide a detailed stability analysis of the proposed approach. Final version of this paper will also include a detailed application to a hypersonic vehicle model to demonstrate the efficacy of the proposed framework.

I. Introduction

Research in adaptive control algorithms for aerospace applications is primarily motivated by the fact that these algorithms have the capability to estimate and suppress the effect of system uncertainties resulting from imperfect aircraft modeling, degraded modes of operation, abrupt changes in aerodynamics, damaged control surfaces, and sensor failures. Although government and industry agree on the potential of these algorithms in providing flight safety and reducing aircraft development costs, a major issue is their poor transient performance. To address this problem, authors of Refs. 1–8 presented modifications to the update law. Specifically, Refs. 1–3 use filtered versions of the control input and state, Refs. 4–6 use a moving time window of the system uncertainty, and Refs. 7, 8 use recorded and instantaneous data concurrently. In contrast to these approaches, Refs. 9–11 present an approach called artificial basis functions that adds modification terms not only to the update law but also to the adaptive controller and show that the system error can be suppressed during the transient system response. Common denominator of the approaches in

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Ref. 1–11 is that they introduce additional mechanisms to model reference adaptive control that capture a form of system uncertainty for suppressing its effect.

Authors have developed and demonstrated a new framework called direct uncertainty minimization for model reference adaptive controllers in [12]. Unlike the approaches in [1–8], the proposed framework consists of a novel architecture involving modification terms in both the adaptive controller and the update law such that these terms are activated when the system error is nonzero and vanishes as the system reaches its steady state. In addition, this new framework directly allows to suppress the effect of system uncertainty on the transient system response through a gradient minimization procedure, and hence, leads to improved system performance. Furthermore, unlike the approaches in Refs. 9–11, the proposed framework is computationally less expensive and it can enforce the system error to approximately stay in a priori given, user-defined error performance bound. The contribution of this paper is to generalize this framework for dynamical systems subject to uncertainty in the control effectiveness and provide a detailed stability analysis of the proposed approach. Final version of this paper will also include a detailed application to a hypersonic vehicle model to demonstrate the efficacy of the proposed framework.

The notation used in this paper is fairly standard. Specifically, \( \mathbb{R} \) denotes the set of real numbers, \( \mathbb{R}^n \) denotes the set of \( n \times 1 \) real column vectors, \( \mathbb{R}^{n \times m} \) denotes the set of \( n \times m \) real matrices, \( \mathbb{R}_+ \) (resp. \( \mathbb{R}_+^n \)) denotes the set of positive (resp. non-negative-definite) real numbers, \( \mathbb{R}_+^{n \times n} \) (resp. \( \mathbb{R}_+^{n \times n} \)) denotes the set of \( n \times n \) positive-definite (resp. non-negative-definite) real matrices, \( \mathbb{S}^{n \times n} \) denotes the set of \( n \times n \) symmetric real matrices, \( \mathbb{ID}^{n \times n} \) denotes the set of \( n \times n \) matrices with diagonal scalar entries, \( (\cdot)^T \) denotes transpose, \( (\cdot)^{-1} \) denotes inverse, \( \text{tr}(\cdot) \) denotes the trace operator, \( ||\cdot||_2 \) denotes the Euclidian norm, \( ||\cdot||_F \) denotes the Frobenius matrix norm, and ‘\( \triangleq \)’ denotes equality by definition. In addition, we write \( \lambda_{\min}(A) \) (resp., \( \lambda_{\max}(A) \)) for the minimum (resp. maximum) eigenvalue of the Hermitian matrix \( A \).

II. Preliminaries on Model Reference Adaptive Control

This section states necessary preliminaries on the model reference adaptive control problem. Consider the uncertain dynamical system given by

\[
\dot{x}_p(t) = A_p x_p(t) + B_p A u(t) + B_p \delta_p(x_p(t)), \quad x_p(0) = x_{p0},
\]

where \( x_p(t) \in \mathbb{R}^{n_p} \) is the state vector available for feedback, \( u(t) \in \mathbb{R}^m \) is the control input restricted to the class of admissible controls consisting of measurable functions, \( \delta_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^m \) is an uncertainty, \( A_p \in \mathbb{R}^{n_p \times n_p} \) is a known system matrix, \( B_p \in \mathbb{R}^{n_p \times m} \) is a known control input matrix, \( \Lambda \in \mathbb{R}_+^{m \times m} \cap \mathbb{ID}^{m \times m} \) is an unknown control effectiveness matrix, and the pair \( (A_p, B_p) \) is controllable.

**Assumption 1.** The uncertainty in (1) is parameterized as

\[
\delta_p(x_p) = W_p^T \sigma_p(x_p), \quad x_p \in \mathbb{R}^{n_p},
\]

where \( W_p \in \mathbb{R}^{s \times m} \) is an unknown weight matrix and \( \sigma_p : \mathbb{R}^{n_p} \rightarrow \mathbb{R}^s \) is a known basis function of the form

\[
\sigma_p(x_p) = [\sigma_{p1}(x_p), \sigma_{p2}(x_p), \ldots, \sigma_{ps}(x_p)]^T.
\]

To address command following, let \( c(t) \in \mathbb{R}^{n_c} \) be a given piecewise continuous command and \( x_c(t) \in \mathbb{R}^{n_c} \).
be the integrator state satisfying

\[ \dot{x}_c(t) = E_p x_p(t) - c(t), \quad x_c(0) = x_{c0}, \quad (3) \]

where \( E_p \in \mathbb{R}^{n_c \times n_p} \) allows to choose a subset of \( x_p(t) \) to follow \( c(t) \). It then follows that (1) and (2) can be augmented as

\[ \dot{x}(t) = A_r x(t) + B\Lambda u(t) + BW_p^T \sigma_p(x_p) + B_r c(t), \quad x(0) = x_0, \quad (4) \]

where \( x(t) \triangleq [x_p^T(t), x_c^T(t)]^T \in \mathbb{R}^n, \) \( n = n_p + n_c \), is the augmented stated vector, \( x_0 = [x_{p0}^T, x_{c0}^T]^T \in \mathbb{R}^n, \)

\[
A \triangleq \begin{bmatrix} A_p & 0_{n_p \times n_c} \\ E_p & 0_{n_c \times n_c} \end{bmatrix} \in \mathbb{R}^{n \times n}, 
\]

\[
B \triangleq [B_p^T, 0_{n_p \times m}]^T \in \mathbb{R}^{n \times m}, 
\]

\[
B_r \triangleq [0_{n_p \times n_c}^T, -I_{n_c \times n_c}]^T \in \mathbb{R}^{n \times n_c}. 
\]

Next, consider the feedback control law given by

\[ u(t) = u_n(t) + u_a(t), \quad (8) \]

where \( u_n(t) \) and \( u_a(t) \) are the nominal feedback control law and the adaptive feedback control law, respectively. Let the nominal feedback control law be given by

\[ u_n(t) = -K x(t), \quad K \in \mathbb{R}^{m \times n}, \quad (9) \]

such that \( A_r \triangleq A - BK \) is Hurwitz. Using (8) and (9) in (4) yields

\[ \dot{x}(t) = A_r x(t) + B_r c(t) + B\Lambda[u_n(t) + W^T \sigma(x(t))], \quad (10) \]

where \( W \triangleq [W_p \Lambda^{-1}, (\Lambda^{-1} - I)] \in \mathbb{R}^{(s+m) \times m} \) is an unknown aggregated weight matrix and \( \sigma(x(t)) \triangleq [\sigma_p^T(x_p(t)), x^T(t)K] \in \mathbb{R}^{(s+m)} \) is a known aggregated basis function. Considering (10), the adaptive control law is given by

\[ u_a(t) = -\hat{W}^T(t)\sigma(x(t)), \quad (11) \]

where \( \hat{W}(t) \in \mathbb{R}^{(s+m) \times m} \) is the estimate of \( W \) satisfying the weight update laws

\[ \dot{\hat{W}}(t) = \gamma \sigma(x(t))e^T(t)PB, \quad \hat{W}(0) = \hat{W}_0, \quad (12) \]

where \( \gamma \in \mathbb{R}_+ \) is the learning rate, \( e(t) \triangleq x(t) - x_r(t) \) is the system error state vector with \( x_r(t) \in \mathbb{R}^n \) being
the reference state vector satisfying the reference

\[ \dot{x}_r(t) = A_r x_r(t) + B_r c(t), \quad x_r(0) = x_{r0}, \]  

(13)

and \( P \in \mathbb{R}_+^{n \times n} \cap S^{n \times n} \) is a solution of the Lyapunov equation

\[ 0 = A_r^T P + PA_r + R, \]  

(14)

where \( R \in \mathbb{R}_+^{n \times n} \cap S^{n \times n} \) can be viewed as an additional learning rate.

Now, using (11) in (10) yields

\[ \dot{x}(t) = A_r x(t) + B_r c(t) - B \Lambda \hat{W}^T(t) \sigma(x(t)), \]  

(15)

and the system error dynamics is given using (13) and (15) as

\[ \dot{e}(t) = A_r e(t) - B \Lambda \hat{W}^T(t) \sigma(x(t)), \quad e(0) = e_0, \]  

(16)

where \( \hat{W}(t) \triangleq \tilde{W}(t) - W \in \mathbb{R}^{(s+m) \times m} \) and \( e_0 \triangleq x_0 - x_{r0} \).

**Remark 1.** The update law given by (12) can be derived using Lyapunov analysis by considering the Lyapunov function candidate

\[ V(e, \tilde{W}) = e^T P e + \gamma^{-1} \text{tr} (\tilde{W} \Lambda^{1/2})^T (\tilde{W} \Lambda^{1/2}). \]  

(17)

Note that \( V(0, 0) = 0 \) and \( V(e, \tilde{W}) > 0 \) for all \( (e, \tilde{W}) \neq (0, 0) \). Now, differentiating (17) yields

\[ \dot{V}(e(t), \tilde{W}(t)) = -e^T(t) R e(t) - 2e^T(t) P B A \hat{W}^T(t) \sigma(x(t)) + 2\gamma^{-1} \text{tr} \, \hat{W}^T(t) \hat{W}(t) \Lambda, \]  

(18)

where using (12) in (18) results in

\[ \dot{V}(e(t), \tilde{W}(t)) = -e^T(t) R e(t) \leq 0, \quad t \in \mathbb{R}_+, \]  

(19)

which guarantees that the system error state vector \( e(t) \) and the weight error \( \tilde{W}(t) \) are Lyapunov stable, and hence, are bounded for all \( t \in \mathbb{R}_+ \). Since \( \sigma(x(t)) \) is bounded for all \( t \in \mathbb{R}_+ \), it follows from (15) that \( \dot{e}(t) \) is bounded, and hence, \( \dot{V}(e(t), \tilde{W}(t)) \) is bounded for all \( t \in \mathbb{R}_+ \). Now, it follows from Barbalat’s lemma [13] that

\[ \lim_{t \to \infty} \dot{V}(e(t), \tilde{W}(t)) = 0, \]  

(20)

which consequently shows that \( e(t) \to 0 \) as \( t \to \infty \).

**Remark 2.** For the case when the nonlinear uncertain dynamical system given by (1) includes bounded exogenous disturbances, measurement noise, and/or the uncertainty in (1) cannot be perfectly parameterized,
then Assumption 1 can be relaxed by considering

$$\delta(t, x) = W^T(t)\sigma(x) + \varepsilon(t, x), \quad x \in D_x,$$

(21)

where $W(t) \in \mathbb{R}^{s \times m}$ is an unknown time-varying weight matrix satisfying $\|W(t)\|_F \leq w$ and $\|\dot{W}(t)\|_F \leq \dot{w}$ with $w \in \mathbb{R}_+$ and $\dot{w} \in \mathbb{R}_+$ being unknown scalars, $\sigma : D_x \rightarrow \mathbb{R}^s$ is a known basis function of the form $\sigma(x) = [1, \sigma_1(x), \sigma_2(x), \ldots, \sigma_{s-1}(x)]^T$, $\varepsilon : \mathbb{R}_+ \times D_x \rightarrow \mathbb{R}^m$ is the system modeling error satisfying $\|\varepsilon(t, x)\|_2 \leq \epsilon$ with $\epsilon \in \mathbb{R}_+$ being an unknown scalar, and $D_x$ is a compact subset of $\mathbb{R}^n$. In this case, the update law given by (12) can be replaced by

$$\dot{\hat{W}}(t) = \gamma \text{Proj}[\hat{W}(t), \sigma(x(t))e^T(t)PB], \quad \hat{W}(0) = \hat{W}_0,$$

(22)

to guarantee the uniform boundedness of the system error state vector $e(t)$ and the weight error $\tilde{W}(t)$, where Proj denotes the projection operator [14].

III. Direct Uncertainty Minimization with Unknown Control Effectiveness

For the model reference adaptive control framework introduced in the previous section, we now develop the direct uncertainty minimization mechanism to improve transient system response. Specifically, we first modify the adaptive feedback control law given by (11) as

$$u_a(t) = -\hat{W}^T(t)\sigma(x(t)) - \phi(t),$$

(23)

where $\phi(t) \in \mathbb{R}^m$ satisfies

$$\phi(t) = \phi(0) + k(B^T B)^{-1}B^T \left[ (e(t) - e(0)) - \int_0^t A_r e(\tau) d\tau \right],$$

(24)

with $k \in \mathbb{R}_+$ being a design parameter. Using (23), the system error dynamics in (16) becomes

$$\dot{e}(t) = A_r e(t) - BA \left[ \hat{W}^T(t)\sigma(x(t)) + \phi(t) \right], \quad e(0) = e_0.$$

(25)

Notice that the ideal system error dynamics has the form

$$\dot{e}(t) = A_r e(t), \quad e(0) = e_0,$$

(26)

under nominal conditions when there is no system uncertainty or control uncertainty, $W \equiv 0$, and $\phi(t) \equiv 0$. Motivating from this standpoint, the mismatch term $\hat{W}^T(t)\sigma(x(t)) + \phi(t)$ in (25) has to be minimized during the transient system response to improve system performance. In the next theorem, we show that (24) allows to achieve this objective through a gradient minimization procedure.

**Theorem 1.** The modification term of the adaptive feedback control law in (24) is the negative gradient
of the cost function given by

\[ J(\cdot) = \frac{k}{2} \left\| \Lambda^{1/2} (\tilde{W}^T(t)\sigma(x(t)) + \phi(t)) \right\|_F^2. \]  

(27)

**Proof.** Negative gradient of the cost function given by (27) with respect to \( \phi(t) \) has the form

\[ -\frac{\partial J(\cdot)}{\partial \phi(t)} = -k \left[ \Lambda (\tilde{W}^T(t)\sigma(x(t)) + \phi(t)) \right], \]

(28)

where (28) can be rewritten using (25) as

\[ -\frac{\partial J(\cdot)}{\partial \phi(t)} = k (B^T B)^{-1} B^T \left[ \dot{e}(t) - \dot{A} r e(t) \right]. \]  

(29)

To construct the modification term of the adaptive feedback control law in (24), let

\[ \dot{\phi}(t) = -\frac{\partial J(\cdot)}{\partial \phi(t)} = k (B^T B)^{-1} B^T \left[ \dot{e}(t) - \dot{A} r e(t) \right], \]

(30)

where (24) is a direct consequence of (30) using integration by parts. \( \blacksquare \)

**Remark 3.** The proposed modification term of the adaptive feedback control law in (24) allows to shape the system error by suppressing the mismatch term \( \tilde{W}^T(t)\sigma(x(t)) + \phi(t) \) in (25) due to gradient minimization, since it is constructed to be the negative gradient of (27) with respect to \( \phi(t) \). Therefore, by adjusting \( k \) in (24), the uncertain dynamical system response and the reference system response can be made close to each other for all time including the transient phase.

Next, we modify the update law given by (12) as

\[ \dot{W}(t) = \gamma \sigma(x(t)) \left[ e^T(t) P B + \xi \phi^T(t) \right], \quad \dot{W}(0) = \tilde{W}_0, \]

(31)

with \( \xi = k/c \) and \( c \in \mathbb{R}_+ \) being a design parameter. The next theorem shows the asymptotic stability of the pair \((e(t), \phi(t))\) as well as the Lyapunov stability of \( \dot{W}(t) \).

**Theorem 2.** Consider the uncertain dynamical system given by (1) subject to Assumption 1, the reference model given by (13), the feedback control law given by (23) with (24), and (31). In addition, let \( \xi \) be chosen such that

\[ \lambda_{\min}(R) - \frac{1}{\xi} \left\| P B \right\|_F^2 \Lambda^* > 0 \]

holds, where \( \| A \|_F \leq \Lambda^* \) (here \( \Lambda^* \in \mathbb{R}_+ \) is a known, possibly conservative bound on the control effectiveness). Then, the solution \((e(t), \phi(t), \dot{W}(t))\) of the closed-loop dynamical system is Lyapunov stable for all initial conditions and \( t \in \mathbb{R}_+ \), \( \lim_{t \to \infty} e(t) = 0 \), and \( \lim_{t \to \infty} \phi(t) = 0 \).

**Proof.** To show Lyapunov stability of the solution \((e(t), \phi(t), \dot{W}(t))\), consider the Lyapunov function
candidate given by
\[
\mathcal{V}(e, \phi, \dot{W}) = e^T Pe + e^{-1} \phi^T \phi + \gamma^{-1} \text{tr} (\dot{W} \Lambda^{1/2})^T (\dot{W} \Lambda^{1/2}).
\] (33)

Note that \(\mathcal{V}(0,0,0) = 0\) and \(\mathcal{V}(e, \phi, \dot{W}) > 0\) for all \((e, \phi, \dot{W}) \neq (0,0,0)\). Differentiating (33) along the closed-loop dynamical system trajectories yields
\[
\dot{\mathcal{V}}(e(t), \phi(t), \dot{W}(t)) \leq -e^T(t)Re(t) - 2\xi \phi^T(t) \Lambda \phi(t) - 2e^T(t)PBA\phi(t),
\]
\[
= -e^T(t)Re(t) - 2\xi \phi^T(t) \Lambda \phi(t) - 2e^T(t)PBA^{1/2}\Lambda^{1/2}\phi(t).
\] (34)

Using Young’s inequality [15] for the last term in (34) gives
\[
-2e^T(t)PBA^{1/2}\Lambda^{1/2}\phi(t) \leq |\mathcal{V}(e(t)PBA^{1/2}\Lambda^{1/2}\phi(t))| \leq \frac{1}{\mu} e^T(t)PBA^{1/2}P\phi(t) + \mu \phi^T(t) \Lambda \phi(t).
\] (35)

Now, setting \(\mu = \xi\) and using (35) in (34) yields
\[
\dot{\mathcal{V}}(e(t), \phi(t), \dot{W}(t)) \leq -e^T(t)Re(t) + \frac{1}{\xi} e^T(t)PBA^{1/2}P\phi(t) - \xi \phi^T(t) \Lambda \phi(t)
\]
\[
\leq -\lambda_{\min}(R)||e(t)||^2 + \frac{1}{\xi} ||e(t)||^2 ||P||^2 \Lambda^* - \xi \lambda_{\min}(\Lambda)||\phi(t)||^2
\]
\[
\leq -||e(t)||^2 \left[ \lambda_{\min}(R) - \frac{1}{\xi} ||P||^2 \Lambda^* \right] - \xi \lambda_{\min}(\Lambda)||\phi(t)||^2
\] (36)

Using the condition (32) in (36), it follows that \(\dot{\mathcal{V}}(e(t), \phi(t), \dot{W}(t)) \leq 0\), which guarantees the Lyapunov stability of the solution \((e(t), \phi(t), \dot{W}(t))\). Since this implies the boundedness of \(e(t), \phi(t), \) and \(\dot{W}(t)\) for all \(t \in \mathbb{R}_+\), it follows from (25) and (30) that \(\dot{e}(t)\) and \(\dot{\phi}(t)\) are bounded for all \(t \in \mathbb{R}_+\), and hence, \(\dot{\mathcal{V}}(e(t), \phi(t), \dot{W}(t))\) is bounded for all \(t \in \mathbb{R}_+\). It now follow from Barbalat’s lemma [13]
\[
\lim_{t \to \infty} \dot{\mathcal{V}}(e(t), \phi(t), \dot{W}(t)) = 0,
\] (37)
which shows that \(\lim_{t \to \infty} e(t) = 0\) and \(\lim_{t \to \infty} \phi(t) = 0\).

**Remark 4.** Since (24) is chosen to minimize the cost function given by (27), \(\phi(t)\) is driven towards the uncertain terms \(\dot{W}^T(t)\sigma(x(t))\). To that end, the modification term in (31) also allows the update law to contribute to this objective.

From a practical standpoint, if \(e(t)\) is sufficiently small, then the design parameter \(\xi\), which affects both modification terms in (23) and (31) can be chosen to be small such that (32) holds. However, as \(e(t)\) becomes large, then \(\xi\) may need to be increased accordingly to put more weight on minimizing the cost function given by (27), and hence, to enforce system error to approximately stay in a priori given, user-defined performance bounds. To achieve this objective, we let \(\xi(t) = k(t)/c\), where \(\xi(t) \in [\xi_{\min}, \xi_{\max}], \xi_{\min} \in \mathbb{R}_+, \xi_{\max} \in \mathbb{R}_+,\) and consider the cost function given by
\[
\mathcal{J}(\cdot) = \frac{k(t)}{2} ||\dot{\Lambda}^{1/2}(\dot{W}^T(t)\sigma(x(t)) + \phi(t))||^2.
\] (38)
Choosing the modification term in (23) as the negative gradient of (38), i.e., 
\[ \dot{\phi}(t) = -\frac{\partial J}{\partial \phi(t)} \], and following similar steps as highlighted in the proof of Theorem 1, it follows by integration by parts that

\[ \phi(t) = \phi(0) + c \left[ \xi(t)(B^T B)^{-1}B^T e(t) - \xi(0)(B^T B)^{-1}B^T e(0) - \int_0^t \dot{\xi}(\tau)(B^T B)^{-1}B^T e(\tau) d\tau \right. 
\]

\[ \left. - \int_0^t \xi(\tau)(B^T B)^{-1}B^T A_r e(\tau) d\tau \right]. \]  

(39)

Notice that in this case the modified update laws become

\[ \dot{\hat{W}}(t) = \gamma \sigma(x(t)) \left[ e^T(t)PB + \xi(t)\phi^T(t) \right], \quad \hat{W}(0) = \hat{W}_0, \]  

(40)

and the condition (32) needs to be replaced with

\[ \lambda_{\min}(R) - \frac{1}{\xi_{\min}} ||PB||_F^2 \Lambda^* > 0, \]  

(41)

where \( ||A||_F \leq \Lambda^* \) (here \( \Lambda^* \in \mathbb{R}_+ \) is a known bound on the control effectiveness). In addition, we choose

\[ \dot{\xi}(t) = -\gamma \xi \left[ f(e)(\xi(t) - \xi_{\min}) + (1 - f(e))(\xi(t) - \xi_{\max}) \right], \quad \xi(0) = \xi_0 \in [\xi_{\min}, \xi_{\max}], \]  

(42)

where \( \gamma \xi \in \mathbb{R}_+ \) and \( f(e) \in [0, 1] \) is a continuously differentiable function such that it is close to 1 when \( e(t) \) is sufficiently small and otherwise close to 0. It follows from (42) that \( \xi(t) \in [\xi_{\min}, \xi_{\max}] \) and \( \xi(t) \) approaches to \( \xi_{\min} \) (resp., \( \xi_{\max} \)) when \( f(e) = 1 \) (resp., \( f(e) = 0 \)). A candidate \( f(e) \) has the form \( f(e) = 1 - \left[ 1 - \text{sech}(c_1 ||e(t)||_P) \right] c_2, \quad ||e(t)||_P \triangleq \sqrt{e^T(t)Pe(t)} \), where it is depicted in Figure 1 for \( c_1 = 5 \) (this is chosen to drive \( \xi(t) \) to \( \xi_{\max} \) if \( ||e(t)||_P \) is larger than 0.5) and \( c_2 = 10 \).

![Figure 1. A candidate f(e) for (42).](image-url)
IV. Application to a Hypersonic Vehicle Model

This section formulates a state space model of a generic hypersonic vehicle (GHV). It then explains how the model is decoupled into longitudinal and lateral dynamics for which separate controllers are designed. The longitudinal and lateral controllers have both a nominal and adaptive portion where the simulation results illustrate both nominal performance, a standard adaptive performance, and the proposed adaptive performance.

For the configuration with an altitude of 80,000 feet and a Mach number of 6, a linearized model under nominal conditions ($\delta_p(x_p(t)) = 0$ and $\Lambda = I$) is obtained in the form of (1) with

$$
A_p = \begin{bmatrix}
-3.70 \times 10^{-3} & -7.17 \times 10^{-1} & 0 & -3.18 \times 10^1 & -2.67 \times 10^{-4} \\
-5.35 \times 10^{-7} & -2.39 \times 10^{-1} & 1 & -2.95 \times 10^{-12} & 2.23 \times 10^{-7} \\
-2.79 \times 10^{-5} & 4.26 & -1.19 \times 10^{-1} & 0 & 3.94 \times 10^{-5} \\
-4.76 \times 10^{-8} & 1.31 \times 10^{-13} & 1 & -4.45 \times 10^{-14} & -1.33 \times 10^{-11} \\
-5.53 \times 10^{-10} & -5.87 \times 10^{3} & 0 & 5.87 \times 10^{3} & 0 \cdots \\
5.99 \times 10^{-16} & -3.14 \times 10^{-11} & 0 & -3.04 \times 10^{-19} & -9.74 \times 10^{-16} \\
1.47 \times 10^{-10} & -4.45 \times 10^{-6} & 0 & 0 & -1.00 \times 10^{-11} \\
-5.29 \times 10^{-12} & 3.98 \times 10^{-8} & 0 & 0 & 1.28 \times 10^{-12} \\
8.08 \times 10^{-28} & 2.04 \times 10^{-22} & 1.01 \times 10^{-20} & 1.17 \times 10^{-16} & -1.73 \times 10^{-31} \\
-8.81 \times 10^{-1} & 0 & 0 & 0 & -1.77 \times 10^{-15} \\
-1.06 \times 10^{-3} & 0 & 0 & -3.18 \times 10^{-21} \\
-1.47 & 0 & 0 & 0 \\
-1.08 \times 10^{-19} & 4.44 \times 10^{-16} & 9.58 \times 10^{-16} & 2.58 \times 10^{-18} \\
\cdots & 0 & 0 & -3.26 \times 10^{-13} \\
-6.97 \times 10^{-2} & -1.04 \times 10^{-2} & -9.99 \times 10^{-1} & -5.35 \times 10^{-3} \\
-1.31 \times 10^{3} & -2.03 & -7.54 \times 10^{-3} & 0 \\
2.07 & -1.55 \times 10^{-3} & -5.31 \times 10^{-2} & 0 \\
-2.38 \times 10^{-4} & 8.54 \times 10^{-1} & -8.84 \times 10^{-3} & -3.00 \times 10^{-6}
\end{bmatrix}
$$

(43)
with the state vector being defined as $x_p(t) = [V(t), \alpha(t), q(t), \theta(t), h(t), \beta(t), p(t), r(t), \phi(t)]^T$, where $V(t)$ denotes the total velocity, $\alpha(t)$ denotes the angle of attack, $q(t)$ denotes the pitch rate, $\theta(t)$ denotes the pitch angle, $h(t)$ denotes the altitude, $\beta(t)$ denotes the sideslip angle, $p(t)$ denotes the roll rate, $r(t)$ denotes the yaw rate, and $\phi(t)$ denotes the roll angle. The control input vector is defined as $u(t) = [\delta_e(t), \delta_a(t), \delta_r(t)]^T$ where $\delta_e(t)$ denotes the elevator deflection, $\delta_a(t)$ denotes the aileron deflection, and $\delta_r(t)$ denotes the rudder deflection. To control the model described above, we decouple the system into its longitudinal and lateral dynamics, design nominal and adaptive controllers for the decoupled system, and then combine the separate controllers to control the overall coupled GHV model (see Figures 2 and 3).

A. Longitudinal Control Design

For the decoupled longitudinal dynamics, we consider the state vector defined as $x_{p_{\text{long}}}(t) = [\alpha(t), q(t)]^T$, with the respective system matrices

$$A_{p_{\text{long}}} = \begin{bmatrix} -2.39 \times 10^{-1} & 1 \\ 4.26 & -1.19 \times 10^{-1} \end{bmatrix},$$

$$B_{p_{\text{long}}} = \begin{bmatrix} -1.33 \times 10^{-4} \\ -1.84 \times 10^{-1} \end{bmatrix}. \tag{45}$$

LQR theory is used to design the nominal controller with $E_{p_{\text{long}}} = [1, 0]$ such that a desired angle of attack command is followed. The controller gain matrix $K_{\text{long}}$ is obtained using the highlighted augmented formulation ((5) and (6)), along with the weighting matrices $Q_{\text{long}} = \text{diag}[20000, 25000, 400000]$ to penalize
\( x_{\text{long}}(t) \) and \( R_{\text{long}} = 12.5 \) to penalize \( u_{\text{long}}(t) \), resulting in the following gain matrix

\[
K_{\text{long}} = \begin{bmatrix}
-1.65 \times 10^2 & -6.09 \times 10^1 & -1.79 \times 10^2 \\
\end{bmatrix}.
\]  

(47)

The solution to \( A_{\text{long}}^T P_{\text{long}} + P_{\text{long}} A_{\text{long}} + R_{1_{\text{long}}} = 0 \), where \( A_{1_{\text{long}}} \triangleq A_{\text{long}} - B_{\text{long}} K_{\text{long}} \), is calculated as

\[
P_{\text{long}} = \begin{bmatrix}
9.64 & 7.14 \times 10^{-1} & 2.07 \times 10^2 \\
7.14 \times 10^{-1} & 1.07 \times 10^{-1} & 1.50 \\
2.07 \times 10^2 & 1.50 & 6.85 \times 10^1 \\
\end{bmatrix},
\]  

(48)

with \( R_{1_{\text{long}}} = \text{diag}[1, 1, 100] \) that is used for both the standard adaptive control design and the proposed controller. For the proposed design, we use (23), (39), (40), and (40), and resort to (42) for enforcing \( \|e_{\text{long}}(t)\|_{P_{\text{long}}} \leq 0.5 \). Additionally, note that \( \xi_{\text{min}} = 10 \) is selected to satisfy (41) and we choose \( c = 2 \). To visualize the overall longitudinal control design, a block diagram is provided in Figure 2.

![Block diagram of separated longitudinal control design](image_url)
B. Lateral Control Design

The decoupled lateral dynamics follow similarly. Specifically, we consider the state vector defined as $x_{\text{lat}}(t) = [\beta(t), p(t), r(t), \phi(t)]^T$, with the respective system matrices

$$A_{\text{lat}} = \begin{bmatrix} -6.97 \times 10^{-2} & -1.04 \times 10^{-2} & -9.99 \times 10^{-1} & -5.35 \times 10^{-3} \\ -1.31 \times 10^{3} & -2.03 & -7.54 \times 10^{-3} & 0 \\ 2.07 & -1.55 \times 10^{-3} & -5.31 \times 10^{-2} & 0 \\ -2.38 \times 10^{-4} & 8.54 \times 10^{-1} & -8.84 \times 10^{-3} & -3.00 \times 10^{-6} \end{bmatrix}, \quad (49)$$

$$B_{\text{lat}} = \begin{bmatrix} -2.47 \times 10^{-5} & -2.18 \times 10^{-4} \\ -8.04 & 10.3 \\ 3.17 \times 10^{-2} & 2.85 \times 10^{-1} \\ 0 & 0 \end{bmatrix}. \quad (50)$$

LQR theory is used to design the nominal controller with

$$E_{\text{lat}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (51)$$

such that a desired sideslip angle command and roll angle command are followed. The controller gain matrix $K_{\text{lat}}$ is obtained using the highlighted augmented formulation along with the weighting matrices $Q_{\text{lat}} = \text{diag}[100, 100, 100, 100, 400000, 2500]$ to penalize $x_{\text{lat}}(t)$ and $R_{\text{lat}} = \text{diag}[1.25, 50]$ to penalize $u_{\text{lat}}(t)$, resulting in the following gain matrix

$$K_{\text{lat}} = \begin{bmatrix} 2.78 \times 10^{2} & -9.08 & -3.62 \times 10^{1} & -3.15 \times 10^{1} & 1.21 \times 10^{2} & -4.37 \times 10^{1} \\ 8.70 \times 10^{1} & 1.52 \times 10^{-1} & -2.72 \times 10^{1} & 1.30 & 8.74 \times 10^{1} & 1.51 \end{bmatrix}. \quad (52)$$
The solution to $A_{\text{lat}}^T P_{\text{lat}} + P_{\text{lat}} A_{\text{lat}} + R_{\text{lat}} = 0$, where $A_{\text{lat}} \triangleq A_{\text{lat}} - B_{\text{lat}} K_{\text{lat}}$ is calculated as

$$P_{\text{lat}} = \begin{bmatrix}
9.87 & -2.22 \times 10^{-3} & -1.15 & 2.00 \times 10^{-1} & 2.10 \times 10^1 & 3.37 \times 10^{-1} \\
-2.22 \times 10^{-3} & 7.18 \times 10^{-3} & -5.92 \times 10^{-4} & 5.89 \times 10^{-2} & 3.12 \times 10^{-3} & 1.35 \times 10^{-1} \\
-1.15 & 5.92 \times 10^{-2} & 2.48 \times 10^{-1} & -8.33 \times 10^{-2} & -2.37 & -1.78 \times 10^{-1} \\
2.00 \times 10^{-1} & 5.89 \times 10^{-2} & -8.33 \times 10^{-2} & 7.40 & 8.97 \times 10^{-1} & 1.53 \times 10^1 \\
2.10 \times 10^1 & 3.12 \times 10^{-3} & -2.37 & 8.97 \times 10^{-1} & 6.88 \times 10^1 & 1.72 \\
3.37 \times 10^{-1} & 1.35 \times 10^{-1} & -1.78 \times 10^{-1} & 1.53 \times 10^1 & 1.72 & 5.81 \times 10^1 \\
\end{bmatrix}, \quad (53)$$

with $R_{\text{lat}} = \text{diag}[1, 1, 1, 1, 100, 100]$ that is used for both the standard adaptive control design and the proposed controller. For the proposed design, we use (23), (39), (40), and (40), and resort to (42) for enforcing $\|e_{\text{lat}}(t)\|_P \leq 0.5$. Additionally, note that $\xi_{\text{min}} = 10$ is selected to satisfy (41) and we choose $c = 2$.

Similar to the previous section, a block diagram is provided in Figure 3 to visualize the control design using the decoupled lateral dynamics to control the overall uncertain system.

Figure 3. Block diagram of separated lateral control design.
C. Nominal System without Uncertainty

The longitudinal and lateral controllers are augmented and applied to the overall coupled system. We first consider the case when there is no uncertainty in the system to show the nominal performance of the control designs. Figure 4 shows the response of the nominal control performance. It can also be seen from this figure that the error signals are not equal to zero which is expected due to the coupling effects.

D. Uncertainty in Control Effectiveness and Stability Derivatives

We now consider the case when the control effectiveness matrix is unknown as well as the stability derivatives $C_{m,\alpha}$ and $C_{n,\beta}$. For this we let $\Lambda = 0.5I$ and we increase $C_{m,\alpha}$ by 500% and decrease $C_{n,\beta}$ by 150%. Figure 5 shows the response with the nominal control, which goes unstable.

A standard adaptive control design is first implemented. For the standard adaptive controllers, we choose the basis functions $\sigma_{\text{long}}(x_{\text{long}}(t)) = [-x_{\text{long}}^T(t)K_{\text{long}}^T, \alpha(t)]^T$ and $\sigma_{\text{lat}}(x_{\text{lat}}(t)) = [-x_{\text{lat}}^T(t)K_{\text{lat}}^T, \beta(t)]^T$ respectively for the longitudinal and lateral controllers. Figures 6 and 7 show the standard adaptive control response. Specifically, Figure 6 shows that for a low learning gain the system transient performance in the sideslip angle and angle of attack is poor. Additionally, the control surface deflection angles exceed practical working limits. To improve the performance, the learning gain is increased as shown in Figure 7. Both the tracking performance and the control response improve; however, as seen in the bottom part of the figure, the standard adaptive controller is unable to enforce a pre-defined bound on the error.

To improve performance further and enforce a user-defined bound on the error, the proposed adaptive controller is then implemented using the same basis functions as the standard adaptive control design. Figures 8 and 9 show the proposed controller performance using the gain varying control. Specifically, Figure 8 illustrates the superior tracking performance and Figure 9 shows the guaranteed bound $\|e(t)\|_P \leq 0.5$ for both the longitudinal and lateral dynamics.

V. Conclusion

To contribute to previous results in model reference adaptive control, we extended the direct uncertainty minimization approach to consider the case in which the control effectiveness is unknown. We showed that the modification terms in the adaptive controller and the update laws allow to suppress the effect of system uncertainty on the transient system response through a gradient minimization procedure for improved system performance and for the system error to approximately stay in a priori given, user-defined error performance bounds. Final version of this paper will also include a detailed application to a hypersonic vehicle model to demonstrate the efficacy of the proposed framework.
Figure 4. Nominal controller performance without uncertainty.

Figure 5. Nominal controller performance with uncertainty in $\Lambda$, $C_{m_{\alpha}}$, and $C_{n_{\beta}}$. 
Figure 6. Standard adaptive controller performance with uncertainty in $\Lambda$, $C_{m\alpha}$, and $C_{n\beta}$ ($\Gamma_{\text{long}} = I_{2 \times 2}$ and $\Gamma_{\text{lat}} = I_{3 \times 3}$).

Figure 7. Standard adaptive controller performance with uncertainty in $\Lambda$, $C_{m\alpha}$, and $C_{n\beta}$ ($\Gamma_{\text{long}} = 100I_{2 \times 2}$ and $\Gamma_{\text{lat}} = 100I_{3 \times 3}$).
Figure 8. Proposed gain varying adaptive control performance with uncertainty in $\Lambda$, $C_{m\alpha}$, and $C_{n\beta}$ ($\Gamma_{\text{long}} = I_{2 \times 2}$ and $\Gamma_{\text{lat}} = \text{diag}[0,1,1,1]$, $\xi_0 = 10$, and $c = 2$).

Figure 9. System error bounds and adaptation gain for Figure 8.

References


