Adaptive Loop Transfer Recovery

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In this paper, a new modification term for use in adaptive control is developed to improve robustness of an existing design. The objective is to recover the loop transfer properties of a reference model associated with a nonadaptive control design. Consequently, this term can increase the level of confidence of an adaptive control system for purposes of increased flight safety. An illustrative example on an unmanned combat aerial vehicle model is provided to illustrate the efficacy of the proposed approach.

I. Introduction

ADAPTIVE control is an attractive approach in nonlinear systems theory due to its ability to cope with system uncertainties and failures. Adaptive controllers can be classified as either indirect or direct. This paper focuses on improving the robustness of a direct adaptive control design. The method is arrived at by examining the loop transfer properties of an adaptive system when linearized about an equilibrium condition. The design approach modifies an adaptive law with the goal of preserving the loop transfer properties of a reference model associated with a nonadaptive control design. The aim is to achieve an adaptive system that preserves the stability margins of a nonadaptive design, while at the same time providing the benefits of adaptation to modeling error.

Many modification terms are reported in the literature [1–11]. Included among these, σ modification [1] adds a pure damping term to the adaptive law, whereas ε modification [2] adds a variable damping term that depends on the error signal. These terms are introduced to ensure that the adapted weights remain bounded. Background learning [3–5] uses current and past data concurrently in the adaptation process. If it allows the adaptation law to continually train in the background based on past data, while still being responsive to dynamic changes based on the current data. In this way, background learning incorporates long-term learning. Q modification [6–8] is similar in spirit to background learning in its intent to improve adaptation performance by using a moving window of the integrated system uncertainty. There is an optimal control theory basis modification term that improves adaptation in the presence of large adaptive gain [9]. More recently, a Kalman filter modification approach [10] has been introduced as a modification term in adaptive control. Kalman filter modification leads to alternative forms for well-known modification terms in adaptive control, such as σ and ε modifications. K modification [11] introduces stiffness to an adaptive law to achieve a prescribed natural frequency and damping ratio.

The adaptive loop recovery (ALR) method described in this paper takes the form of a modification term. ALR can be applied to improve robustness of an adaptive design and can be employed in combination with σ, ε, as well as other modification terms. The setting for our presentation will be in the context of adaptive augmentation of an existing linear control design. The goal is to design an adaptive controller so that the state of an uncertain system follows the state of a reference model, while at the same time preserving, to the extent possible, the reference model stability margins. The margins of interest are gain margin and time delay margin. Stability analysis of adaptive control with ALR modification is performed by examining its asymptotic properties using known results from singular perturbation theory [12–14]. It is shown that the boundary layer dynamics are exponentially stable to a slow manifold within which the loop properties of the reference model are preserved. Furthermore, it is shown that increasing the ALR gain does not amplify the effect of sensor noise. We provide results for an example that illustrates augmentation of an existing unmanned combat aerial vehicle (UCAV) flight control system design taken from the literature [15]. These results illustrate the improved margins attainable through the introduction of ALR modification. Notationally, ∥ · ∥ is used to denote a vector norm. Unless otherwise stated, ∥ · ∥ is used to denote an induced matrix norm, and β'(x) is used to denote the derivative of β(x) with respect to x.

The organization of the paper is as follows. Section II provides preliminaries needed for the rest of the paper. Section III describes the main concept of ALR modification, including a theorem regarding a stability property that can be proven for this approach. Section IV addresses the issue of sensor noise. The UCAV application is given in Sec. V. Conclusions are summarized in Sec. VI.

II. Preliminaries

We begin by presenting a simple formulation of the model reference adaptive control problem. The presentation in this section is in the context of a simplified adaptive control problem formulation and is not intended to imply that ALR modification is limited by this formulation. Extensions required to apply ALR modification in more complex problem formulations are discussed in Sec. III and in greater detail in the application treated in Sec. V.

Consider the controlled uncertain system given by

$$\dot{x}(t) = Ax(t) + B[u(t) + Δ(x(t))]$$

where $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input; $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are known matrices; and $Δ(x) : \mathbb{R}^n \to \mathbb{R}^m$ is a matched uncertainty that is continuously differentiable with respect to $x$. Furthermore, we assume that the pair $(A, B)$ is controllable, the full state is available for feedback, and the control input is restricted to the class of admissible controls consisting of measurable functions.

In flight control design, the matrices $(A, B)$ in Eq. (1) are usually obtained by linearizing the equations of motion at selected flight conditions, and the resulting set of linear models are used to design a gain-scheduled flight controller. It is assumed that a nominal linear controller for the system in Eq. (1) exists at each flight condition and can be written in the form

$$u_0 = -K, x(t) + K_r r(t)$$

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where \( r(t) \in \mathbb{R}^r \) is the bounded piecewise continuous reference command, \( K_e \in \mathbb{R}^{n \times n} \) is the state gain matrix, and \( K_r \in \mathbb{R}^{m \times r} \) is the input gain matrix, where \( r \leq m \). It should be noted that existing flight control designs commonly contain dynamics, in which case one can augment the controller dynamics with the dynamics in Eq. (1), consider an expanded state made up of the aircraft states and the controller states, and rewrite the dynamics and controller in a form similar to that of Eqs. (1) and (2). Hence, there is no loss in generality with respect to dynamic controllers. However, to further simplify the discussion, we introduce the following assumption.

**Assumption 1:** The uncertainty in Eq. (1) can be linearly parameterized using a set of basis functions in the form

\[
\Delta(x) = W^T \beta(x)
\]  

(3)

where \( W \in \mathbb{R}^{q \times m} \) is the unknown, constant, and bounded weight matrix, and \( \beta(x) : \mathbb{R}^n \rightarrow \mathbb{R}^q \) is the bounded piecewise continuous reference \( (8) \). The uncertainty in Eq. (1) can be linearized as

\[
\dot{x}_m(t) = A_m x_m(t) + B_m r(t), \quad A_m = A - BK_r, \quad B_m = BK_e
\]  

(4)

Then, an augmenting adaptive controller can be defined in the form

\[
u(t) = u_a(t) + u_a(t), \quad u_a = \hat{W}^T(t) \beta(x(t))
\]  

(5)

Defining \( e(t) \equiv x(t) - x_m(t) \), it is well known \([1,2]\) that the adaptive law

\[
\dot{\hat{W}}(t) = \gamma \beta(x(t)) e^T(t) P B, \quad \gamma > 0
\]  

(6)

where \( P \in \mathbb{R}^{n \times n} \) is a positive definite solution of the Lyapunov equation

\[
0 = A_m^T P + PA_m + Q, \quad Q > 0
\]  

(7)

ensures \( \hat{W}(t) \) remains bounded and \( e(t) \rightarrow 0 \) as \( t \rightarrow \infty \). The resulting adaptive control system is illustrated in Fig. 1. In the case where there are bounded external disturbances or when Assumption 1 is relaxed to

\[
|\Delta(x) - W^T \beta(x)| < \epsilon(x), \quad \epsilon(x) < \bar{\epsilon}
\]  

(8)

it can be shown that \( e(t) \) is uniformly ultimately bounded. Furthermore, in this case, it is common practice to modify the adaptive law in Eq. (6), using either \( \sigma \) modification \([1]\), \( e \) modification \([2]\), and/or parameter projection \([16]\) to ensure that \( \hat{W}(t) \) remains bounded. Other modifications have been employed to improve adaptation efficiency \([3,11] \). These can be collectively categorized as methods of composite adaptation \([17] \).

In the next section, we introduce a novel modification to the standard adaptive control law in Eq. (6) that is motivated from the perspective of ensuring that the controller seeks to maintain the loop transfer properties of the reference model. The objective is to maintain both the tracking performance of the reference model and its stability margin properties, to the extent possible, even in the presence of uncertainty.

### III. Adaptive Loop Transfer Recovery Modification

From the perspective of verification and validation of adaptive flight control systems, we can formulate and attempt to address the following two problems:

**Problem 1:** Design an adaptive controller so that \( x(t) \) asymptotically follows \( x_m(t) \) while preserving the margins of the reference model in the absence of uncertainty.

**Problem 2:** In addition, preserve the reference model margins to the degree possible, even under uncertainty and/or failures in actuation.

#### A. Loop Transfer Properties of Adaptive Systems

Although the aspect of tracking performance can be dealt with in a nonlinear setting by employing Lyapunov stability analysis, addressing issues related to margins requires linearization. Within the context of the adaptive control problem formulated in the previous section, this reduces to linearizing the adaptive law in Eq. (6) about an equilibrium point \( \hat{x} \) for the system defined by Eqs. (4), (5), and (6) when \( r(t) \) is constant.

Figure 2 illustrates the result of this linearization with weights frozen. The upper portion of this drawing represents the closed-loop system without uncertainty. The margins calculated with the loop broken at \( \times \) in this drawing with \( \hat{W} = 0 \) correspond to the margins of the reference model. However, this picture is fallacious, for in reality the weights of the reference model are not frozen. But even if \( e(t) \) is zero and \( \hat{W}(t) \) is constant, it is apparent that the margins of the reference model are not maintained; instead, they are modified in an unknown way by the lower feedback block. In the case of varying weights, it is not possible to calculate margins on the basis of
Fig. 2 because the lower portion of this diagram is a time-varying matrix block with \( \bar{W} \) replaced by \( \hat{W}(t) \). However, Fig. 2 is useful in that it can be argued that it might still be possible to solve Problems 1 and 2 if we can approximately enforce the constraint

\[
\hat{W}(t) \frac{df(x(t))}{dt(t)} = 0 \tag{9}
\]

in the adaptive process by introducing a modification term that in the limit satisfies Eq. (9) for all \( t > 0 \) as the modification gain approaches infinity. In this direction, we introduce the following assumption, with minimal loss of generality.

**Assumption 2: \( \hat{\beta}_i(x) \) has full column rank.**

**Remark 1:** The basis functions should be chosen with sufficient redundancy to allow approximate satisfaction of Eq. (9) while at the same time retaining the property that \( \hat{W}(t) \beta(x(t)) \) approximately cancels \( \Delta_e(x(t)) \). One way to ensure this is to include a bias term in the basis vector. For example, suppose \( \Delta(x) = N x \), where \( N \) is a constant but unknown matrix. Then \( \hat{\beta}(x) = x \) satisfies Assumption 1, but the condition in Eq. (9), if satisfied exactly, would then imply \( \hat{W} = 0 \). However, if \( \hat{\beta}(x) = [1, x^T]^T \) and we denote the first column of \( \hat{W}(t) \) by \( \hat{W}_1(t) \) and the remaining columns by \( \hat{W}_j(t) \), then \( \hat{W}_1(t) = N x(t) \) and \( \hat{W}_j(t) = 0 \) would satisfy both Eq. (9) and \( \hat{W}(t) \beta(x(t)) = \Delta_e(x(t)) \). Moreover, \( \hat{\beta}_i(x) \) has full column rank.

**Remark 2:** To ensure recovery of the loop properties of the reference model, it is necessary that the condition in Eq. (9) be maintained during both transitory and under steady state conditions. The rank condition on \( \hat{\beta}_i(x) \) in Assumption 2 is required to ensure that this occurs. This will become apparent later in the stability analysis.

**B. Adaptive Loop Recovery**

Approximate enforcement of the constraint in Eq. (9) can be achieved in the adaptive law through a loop transfer recovery approach described in this section. If

\[
\mathcal{J}(t) = \| \hat{W}(t) \beta(x(t)) \|^2 / 2 \tag{10}
\]

where \( \| \cdot \|_F \) denotes the Frobenius norm, then

\[
\frac{d}{dt}\mathcal{J}(t) = \beta(x(t)) \hat{W}_t(t) \hat{W}(t) \tag{11}
\]

So, in the spirit of other modification terms that have been employed in the past, we use the negative of the gradient in Eq. (11) to modify the adaptive law in Eq. (6) to the following:

\[
\hat{W}(t) = \gamma \beta(x(t)) e^T(t) P B \hat{W}(t) - k_w \beta(x(t)) \beta_t(x(t)) \hat{W}(t), \quad k_w > 0 \tag{12}
\]

where the second bracketed term is the ALR modification, and \( k_w \) is the ALR modification gain. The ALR procedure consists of selecting \( k_w \) sufficiently large so that \( \mathcal{J}(t) \) in Eq. (10) remains sufficiently small for all \( t > t_1 > 0 \), where \( t_1 \) is also made sufficiently small by choosing \( k_w \) sufficiently large. To ensure ultimate boundedness under the assumption that the relaxed condition in Eq. (8) is satisfied, the adaptive law in Eq. (12) can be further modified to include an additional modification term, such as \( e \) modification [2]

\[
\hat{W}(t) = \gamma \beta(x(t)) e^T(t) P B \hat{W}(t) - k_w \beta(x(t)) \beta_t(x(t)) \hat{W}(t) - k_e e^T(t) P B \hat{W}(t), \quad k_e > 0 \tag{13}
\]

**Remark 3:** The notion of ‘sufficiently large’ in the context of the singular perturbation analysis means that \( k_e \gg 1 \). However, for this to be the case, one needs to take into account the relative magnitude of the terms in Eq. (12). In the stability analysis, it is assumed that the ALR modification term is normalized so that its magnitude relative to the first term is the same order of magnitude when \( k_e = 1 \).

**Theorem 1:** Consider the uncertain dynamical system (1), with the control law (5), together with the adaptive law (12) subject to Assumptions 1 and 2. There exists \( k_w > k_w \) such that for all \( k_w > k_w \), \( x(t) \) asymptotically tracks \( x_{\text{ref}}(t) \) and \( \hat{W}(t) \) remains bounded. Furthermore, \( \exists k_e > 0 \) and \( \sigma > 0 \) such that \( \| W(t) \|_2 x_e(t) \| \leq k_e e^{-\sigma t} + O(1/k_w) \).

**Proof:** Using Assumption 1, the system dynamics in Eq. (1) can be expressed as

\[
\dot{x}(t) = A x(t) + B u(t) + W^T \beta(x(t)) \tag{14}
\]

Using Eqs. (2) and (5), the closed loop dynamics can be written as

\[
\dot{x}(t) = A_w x(t) + B_w r(t) - B W^T(t) \beta(x(t)) \tag{15}
\]

where \( W(t) = W(t) - W(t) \). The error dynamics can be written as

\[
\dot{e}(t) = A_w e(t) - B W^T(t) \beta(x(t)) \tag{16}
\]

From Eq. (12), we can write the dynamics of \( W(t) \) in the form

\[
\dot{W}(t) = e \gamma \beta(x(t)) e^T(t) P B - \gamma \beta(x(t)) \beta_t(x(t)) \hat{W}(t) \tag{17}
\]

where \( e \equiv 1/k_w \).

Equations (16) and (17) have the form of a singularly perturbed system with \( e \) viewed as a small parameter [12–14,18]. According to Tikhonov’s theorem [18], an approximation to the solution for \( e(t, \varepsilon) \) can be constructed from the solution of the reduced system given by

\[
\dot{\hat{W}} = e \gamma \beta(x(t)) e^T(t) P B - e \gamma \beta(x(t)) \beta_t(x(t)) \hat{W}(t) \tag{18}
\]

where \( \hat{W} \) is an isolated root of

\[
\dot{\hat{W}} = e \gamma \beta(x(t)) e^T(t) P B - e \gamma \beta(x(t)) \beta_t(x(t)) \hat{W} = 0 \tag{19}
\]

Equations (18) and (19) follow directly from Eqs. (16) and (17) after setting \( e = 0 \) in Eq. (17). A requirement for the application of Tikhonov’s theorem is that the equilibrium point of the boundary layer system is exponentially stable uniformly in \( x(t) \). The boundary layer system in this case is defined by applying the time transformation \( \tau = t/e \) to Eq. (17) and again setting \( e = 0 \),

\[
d \hat{W}_b(t)/d\tau = -e \gamma \beta(x(t)) \beta_t(x(t)) \hat{W}_b(t), \quad \hat{W}_b(0) = \hat{W}(0) \tag{20}
\]

with \( r \) and therefore \( x(t) \) regarded as a fixed parameter. The main result of Tikhonov’s theorem is that

\[
e(t, \varepsilon) = e(t) + O(e) \tag{21}
\]

over any time interval \( t \in [t_1, t_2] \) for which a solution to Eqs. (16) and (17) exists. In Eq. (21), \( O(e) \) signifies a term that is of order \( e \), meaning that there exists constants \( C \) and \( C^* \) such that \( |O(e)| < C e \) whenever \( 0 < e < e^* \). The solution to the boundary layer system (20) can also be used to construct a composite approximation for \( W(t) \) over the same time interval:

\[
\hat{W}(t) = \hat{W}_b(t/e) + \hat{W}(t) - \hat{W}(0) + O(e) \tag{22}
\]

Recognizing that \( A_w(t) \beta(x(t)) \beta_t(x(t)) \geq 0 \) and the fact that \( \beta_t(x(t)) \) has full column rank (Assumption 2), it follows that the solutions of Eq. (20) are exponentially bounded and approach a manifold, within which \( \hat{W}(t) \) satisfies Eq. (19). However, the roots of Eq. (19) are not isolated. The significance of this fact is that one cannot use Eq. (19) to solve for \( \hat{W}(t) = \beta(x(t)) \) and substitute the result into Eq. (18) to uniquely construct a solution for \( e(t) \).

We next show that the requirement for an isolated root does not impose an absolute impediment to stability analysis. To see this, decompose \( \hat{W}(t) \) into

\[
\hat{W}(t) = \hat{W}_b(t) + \hat{W}_b(t) \quad (23)
\]
with the property that the columns of $\hat{W}_t(t)$ lie in the null space of $\beta^T_t(x(t))$ and the columns of $\hat{W}_t(t)$ lie in the range space of $\beta_t(x(t))$. Since these are complementary and orthogonal subspaces, we can equivalently express the singularly perturbed system in Eqs. (16) and (17) as

$$\dot{\hat{\beta}}(t) = A_0\hat{\beta}(t) - B\hat{W}_t(t)\hat{\beta}(t)$$

where $A_0 = \beta_t(x(t))\beta^T_t(x(t))\beta_t(x(t))^{-1}\beta^T_t(x(t))$, and $[I - A_0(t)]$ is a time dependent projection operator. Setting $\varepsilon$ to zero produces a condition similar to Eq. (19), except that it applies to $\hat{W}_t(t)$, and since $\hat{W}_t(t)$ lies in the range space of $\beta_t(x)$, it follows that there is a unique root at the origin. Furthermore, applying the time transformation $\tau = t/\varepsilon$ and setting $\varepsilon$ to zero produces boundary layer dynamics similar to Eq. (20), except that it applies to $\hat{W}_t(t)$. Because $\beta_t(x)\beta^T_t(x) \geq 0$ and $\hat{W}_t(t)$ evolves in the range space of $\beta_t(x)$, it follows that $\hat{W}_t(t)$ is exponentially bounded. Thus, Assumption 2 ensures that the boundary layer system associated with Eq. (24) has a unique exponentially stable equilibrium point at the origin. Also, $\hat{W}_t(t)$ describes the behavior of $\hat{W}(t)$ within the slow manifold [18]. The reduced system corresponding to Eq. (24) is:

$$\dot{\hat{e}}(t) = A_0\hat{e}(t) - B\hat{W}_t(t)\hat{\beta}(t)$$

where $A_0 = \beta_t(x(t))\beta^T_t(x(t))\beta_t(x(t))^{-1}\beta^T_t(x(t))$, and $[I - A_0]$ is a symmetric matrix. Setting $\varepsilon \to 0$ and $\hat{W}_t(t)$ approaches a finite limit. Consider the positive definite Lyapunov function candidate for the reduced system in Eq. (25):

$$V(e, \hat{W}_t) = e^T(t)P e(t) + \text{tr}([\hat{W}(t)]P)$$

Differentiating Eq. (26) with respect to time and using the expressions in Eq. (25) leads to

$$\dot{V}(e, \hat{W}_t) = -e^T(t)Qe(t) - 2e^T(t)PB\hat{W}_t A_0\hat{e}(t)$$

Since the columns of $\hat{W}_t(t)$ lie in the null space of $\beta^T_t(x(t))$, it follows that $\hat{W}_t(t)A_0 = 0$, and therefore $V(e, \hat{W}_t(t)) \leq 0$. Hence, $e(t)$ and $\hat{W}_t(t)$ are bounded. Since $\hat{W}_t(t)$ is lower-bounded and $\dot{V}(e, \hat{W}_t(t)) \leq 0$, $\dot{V}(e, \hat{W}_t(t))$ approaches a finite limit as $t \to \infty$. Moreover, $\dot{V}(e, \hat{W}_t(t))$ is a uniformly continuous function of time, because its time derivative $\dot{V}(e, \hat{W}_t(t))$ is bounded, which is a consequence of the fact that $e(t)$ and $\hat{W}_t(t)$ are bounded. Hence, from Barbalat’s lemma [18], $\dot{V}(e, \hat{W}_t(t))$ asymptotically goes to zero, which implies that $\lim_{t \to \infty} e(t) = 0$. Moreover, $\lim_{t \to \infty} \hat{W}_t(t) = \hat{W}$ exists.

Since the reduced system in Eq. (25) has an exponentially stable equilibrium point and the boundary layer system associated with Eq. (24) is exponentially stable at the origin, it follows from Theorem 11.4 in [18] that there exists an $\varepsilon^*$ such that, for all $\varepsilon < \varepsilon^*$, the full order system in Eq. (24) is exponentially stable and $\lim_{t \to \infty} e(t) = 0$. Because $\hat{W}_t(t)$ is exponentially bounded and $\hat{W}_t(t)$ lies in the null space of $\beta^T_t(x(t))$, it follows from the application of Eq. (22) to $\hat{W}_t(t)$ that, for sufficiently large values of $k_w$, $\|\hat{W}^T(t)\hat{\beta}(x(t))\|$ is exponentially bounded within $O(\varepsilon)$, which completes the proof of Theorem 1.

Remark 4: When this result is extended to the case wherein the condition in Eq. (3) is relaxed to the condition in Eq. (8), or to situations in which the parameterization applies on a bounded set, it can be shown that $e(t)$ is uniformly ultimately bounded and that $\|\hat{W}^T(t)\hat{\beta}(x(t))\|$ approaches a $O(1/k_w)$ neighborhood of the origin, with the introduction of an appropriate modification term, such as in Eq. (13).

Remark 5: Extensions to nonlinear parameterizations of $\Delta(t, x)$, such as those encountered using multilayer neural networks, are also possible by computing the gradient in Eq. (11) with respect to only the output layer weights and then applying the result as an ALR modification term to the adaptive law for only the output layer weights, in the same manner as in Eq. (13). In this context, $\hat{\beta}(x(t))$ in Eq. (11) is viewed as dependent on all the input and internal layer weights, and the adaptive laws that pertain to those weights are not modified by ALR. Then, the proof of Theorem 1 can easily be extended to include, for example, adaptive laws for nonlinear parameterizations of uncertainty of the form in [19] (see the example in Sec. V of this paper). Therefore, ALR can be applied in a complementary manner to many existing methods of adaptive control design.

IV. Effect of Sensor Noise

Since ALR relies on the asymptotic properties of the error system in Eqs. (16) and (17), it is important to take into account the effect of sensor noise. In particular, we are concerned here with the possibility that the effect of sensor noise on $u_{t}(t)$ in Eq. (5) may be amplified as the ALR gain $k_w$ in Eq. (12) is increased. If we view sensor noise $\nu(t)$ as additive to the feedback of $x(t)$, then Eqs. (5) and (12) become

$$u(t) = u_{t}(t) + \nu(t), \quad u_{t} = \hat{W}^T(t)\hat{\beta}(\hat{x}(t))$$

$$\dot{\hat{W}}(t) = \gamma(\hat{\beta}(\hat{x}(t))\hat{\beta}^T(t)PB - k_w\hat{\beta}(\hat{x}(t))\hat{\beta}^T(t)\hat{W}(t))$$

$$k_w > 0$$

where $\hat{x}(t) \equiv x(t) + \nu(t)$ and $\hat{x}(t) \equiv \hat{x}(t) - x_{\nu}(t)$. It can be seen that sensor noise enters $u_{t}(t)$ directly through the basis vector and indirectly through the effect it has on $\hat{W}(t)$. However, $k_w$ only affects the latter, and superficially it appears that, as $k_w$ increases, its effect on $\hat{W}(t)$ is amplified. On the other hand, the purpose of increasing $k_w$ in the first place is to reduce the size of $\hat{\beta}(\hat{x}(t))\hat{W}(t)$. Thus the net effect is not evident from a simple examination of these equations.

Our principle concern here is the effect that $k_w$ has when it is large. Under this condition, we can make further use of the singular perturbation analysis in Sec. III and consider the effect that noise has on the components of $\hat{W}(t)$ as defined in Eq. (23). It has been shown that the motion in the slow manifold satisfies Eq. (25). Therefore, the slow motion with sensor noise will be similar but with $x(t)$ replaced by $x(t) + \nu(t)$ in the right-hand side of Eq. (25), which is independent of $k_w$. The motion in the boundary layer satisfies the third equation in Eq. (24). Substituting $t = t/\varepsilon$ and letting $\varepsilon \to 0$, we obtain the following boundary layer dynamics:

$$d\hat{W}_{t}(t)/dt = \gamma(\hat{\beta}(x(0) + \nu(t))\hat{\beta}^T(x(0) + \nu(t))\hat{W}_{t}(t))$$

which is exponentially attracted to the origin. Since $t = t/\varepsilon$, it follows that the duration of this transient on the $t$ time scale is inversely proportional to $k_w$, and its amplitude is independent of $k_w$.

The composite behavior for $\hat{W}(t)$ is obtained using Eq. (22):

$$\hat{W}(t) = \hat{W}_{t}(t/\varepsilon) + \hat{W}_0(t) - \hat{W}_0(0) + O(\varepsilon)$$

Therefore, for $\varepsilon$ sufficiently small ($k_w$ sufficiently large), the value of $k_w$ has a negligible effect on the manner in which sensor noise enters $u_{t}(t)$. This property is illustrated numerically in the next section.
V. Application to Adaptive Control of an Unmanned Combat Aerial Vehicle

This section illustrates the effectiveness of ALR modification for a complex aircraft flight control example, based on an adaptive control design for the Boeing UCAV[15]. UCAV is a tailless configuration with three elevon controls on each wing, plus thrust vectoring for yaw control. A control allocator is used to distribute three virtual controller commands for the roll, pitch, and yaw axes to the seven available actuators. This provides an opportunity to exploit control redundancy in the event of a single or even multiple control surface failures. However, the nominal control design does not fully exploit this redundancy, so the nominal design has been augmented with an adaptive element as described in [15], which essentially follows the approach described in Sec. II but with some modifications that are described in the next section.

A. Problem Formulation

The linearized UCAV plant dynamic has redundant actuation and takes the form

\[ \dot{x}_p(t) = A_p x_p(t) + B_{p,u} A(G u(t) - u_i(t)) + B_{p,y} \Delta x_p(t) + B_{p,x} y(t) \]

\[ y(t) = C_{p,x} x_p(t) + D_{p} A(G u(t) - u_i(t)) + \Delta x_p(t) \]  

(31)

where \( x_p(t) \) is the state vector that consists of angle of attack \( \alpha(t) \), angle of sideslip \( \beta(t) \), body roll rate \( p(t) \), body pitch rate \( q(t) \), and body yaw rate \( r(t) \) in \( \mathbb{R}^5 \) is the state vector that consists of angle of attack \( \alpha(t) \), angle of sideslip \( \beta(t) \), body roll rate \( p(t) \), body pitch rate \( q(t) \), and body yaw rate \( r(t) \). \( u_i(t) = u(t) - G u(t) - \Delta u(t) \) is the control input, and \( G \in \mathbb{R}^{7 \times 5} \) is a control allocation matrix. \( y(t) = [A_i(t), \beta(t), p(t), r(t)] \) is used to define the regulated outputs, where \( A_i(t) \) is the sensed acceleration at the c.g. normal to the body axis, and \( y_i(t) \) is the command vector corresponding to \( y(t) \). In Eq. (31), \( A \in \mathbb{R}^{7 \times 7} \) is nominally an identity matrix, and a control failure can be represented by setting one or more of its diagonal elements \( (A_i)_{ij} \) to zero. Note from the manner in which \( u_i(t) \) appears in Eq. (31) that the augmenting adaptive controller element has direct access to each actuator. Also note that there is a direct feedback term in \( y(t) \) due to the fact that one of the regulated outputs is acceleration. The nominal controller is defined using a dynamic model of the form

\[ \dot{x}_c(t) = A_c x_c(t) + B_{c,p,y} \dot{x}_p(t) + B_{c,y} y(t) \]

\[ u(t) = -K_c x_c(t) + K_{c,x} x_c(t) + K_{c,y} y(t) \]  

(32)

where \( x_c(t) = [q(t), p(t), r(t), r_e(t)] \) is the controller state vector that consists of pitch integrator state \( q(t) \), roll integrator state \( p(t) \), yaw integrator state \( r(t) \), and a yaw rate washout filter state \( r_e(t) \). By augmenting these dynamics to the plant model and defining \( \hat{x}(t) = [x(t), x_c(t)] \), it is possible to write Eqs. (31) and (32) in a form similar to Eqs. (1) and (2).

An actuator is considered redundant if its failure does not change the range space of \( B_{p,u} \). This is approximately true in aircraft applications where the means of the control is primarily through moment generation. Because in this case there are seven independent control surfaces and only three moment axes, it is conceivable that up to four control surfaces could fail with minimal change in the column space of \( B_{p,u} \), insofar as the rows of \( B_{p,u} \) corresponding to \( \dot{p}, \dot{q}, \) and \( \dot{r} \) equations are concerned. Assuming this to be the case, then it can be shown that the total uncertainty in Eq. (31) can be approximately represented by

\[ \dot{x}(t) = A x(t) + B_A [G u(t) - u_i(t) + \Delta x(t), y_i(t)] + B_{c,x} x_c(t) \]

(33)

This implies that the basis function vector in Assumption 1 should depend on \( z(t) = [x^T(t), y_c^T(t)] \), but the condition in Assumption 2 applies only to \( x(t) \). The proof of Theorem 1 can be extended to this case by modifying the Lyapunov function candidate used in Eq. (26)

\[ V(e(t), \hat{W}_0(t)) = e^T(t) P e(t) + \text{tr}[\hat{W}_0^T(t) \hat{W}_0(t) (A + \delta I_n)] \]

(34)

where \( \delta > 0 \) can be made arbitrarily small. Using Eq. (34) in place of Eq. (26), it can be shown that \( e(t) \) is uniformly ultimately bounded to an \( O(\delta) \) neighborhood of the origin.

B. Numerical Results

For this example we employed a multilayer neural network of the form

\[ \beta(z, V(t)) = \sigma(V(t)z) \]  

(35)

to define the basis functions, where \( V(t) \in \mathbb{R}^{(n+1) \times q} \), \( z = [x^T, r^T] \), \( \sigma(a) = [1, x^T(a)]^T \), and \( s(a) \) is a column vector of sigmoidal functions having elements defined by

\[ s_i(a) = 1/(1 + e^{-a_i}), \quad i = 1, \ldots, q \]  

(36)

where \( a_i > 0 \) are the activation potentials. The multilayer weight adaptive laws [19] with e-modification [2] are given by:

\[ \dot{\hat{W}}(t) = -\gamma_1 [s_i(a_i) \hat{V}(t) z(t)] e^T(t) PB \]

\[ + k_x \hat{e}(t) PB [\hat{W}(t)] \]  

(37)

\[ \hat{V}(t) = -\gamma_2 z^T(t) e(t) PB [\hat{W}(t)] s_e(t) e(t) PB \]

\[ + k_e \hat{e}(t) PB \hat{V}(t) / 2 \]  

(38)

where \( s_e(t) \) is a column vector whose first element is zero, and the remaining elements are \( d_e(t) \) uniformly between 0.1 and 1.0 and used \( k_e = 1 \) for the e modification gain. Also, all the elements of \( z(t) \) and \( r(t) \) were normalized so that they have the same relative magnitude.

Table 1 Nominal controller margins with no failures

<table>
<thead>
<tr>
<th>Loop</th>
<th>Gain margin, dB</th>
<th>Phase margin, deg</th>
<th>TD margin, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOE</td>
<td>Inf</td>
<td>Inf</td>
<td>Inf</td>
</tr>
<tr>
<td>LME</td>
<td>Inf</td>
<td>Inf</td>
<td>Inf</td>
</tr>
<tr>
<td>LIE</td>
<td>Inf</td>
<td>121.1</td>
<td>0.58</td>
</tr>
<tr>
<td>RIF</td>
<td>Inf</td>
<td>121.8</td>
<td>0.57</td>
</tr>
<tr>
<td>RME</td>
<td>21.6</td>
<td>61.9</td>
<td>0.46</td>
</tr>
<tr>
<td>ROE</td>
<td>Inf</td>
<td>84.6</td>
<td>0.21</td>
</tr>
<tr>
<td>TVC</td>
<td>Inf</td>
<td>Inf</td>
<td>Inf</td>
</tr>
</tbody>
</table>

*Margins are computed with each control loop broken one at a time.
The parameter values used to define the dynamics and nominal controller design in Eqs. (32) and (33) were taken from [15] (Appendix A). The actuators were modeled as first-order systems with a time constant of 0.01 s and a rate limit of 110 deg/s. The measurements are sampled at 100 Hz and the step size used to integrate the adaptive law was set to 0.001 s.

Figure 3 shows the performance of the nominal controller for a sequence of commands in $A_z$, $p$, and $r$. The nominal controller margins are shown in Table 1, where TVC denotes thrust vector control in the yaw axis; LOE (LME/LIE) denotes the left outboard (midboard/inboard) elevon, and a similar pattern of notation is used for the right-side elevons (ROE). These margins were obtained by breaking the control loops one at a time at each actuator input. All the nominal margins are excellent. However, Fig. 4 shows that the performance of the nominal controller is seriously degraded when an ROE failure is injected at $t = 5$ s. The reason for this poor performance can be seen in Table 2. Note that, under this failure, the phase margin for the RME control loop is reduced to 8.4 deg. This shows that, from the perspective of robustness under failure, the nominal controller design does not make effective use of the available control redundancy. One approach to solving this problem might be to reconfigure the flight control system for this failure, but this presumes that the failure is detected and correctly identified. Therefore, an alternative approach that employs adaptation is investigated below.

Figure 5 shows the performance of the adaptive controller under ROE failure, with an adaptation gain of $k_w = 100$. This controller includes an $\epsilon$ modification term but without ALR modification ($k_w = 0$). Figures 6 and 7 show what happens under ROE failure as $\gamma$ is increased to 200 and 500, respectively. Clearly, there is a very wide range of satisfactory choices for the adaptation gain, and a reasonable choice is $\gamma = 200$.

To assess the margin of the adaptive controller, we introduced time delay in the RME control channel. According to Table 2, with the nominal controller, this channel can tolerate up to 0.46 s of delay before the system goes unstable when there is no failure. Under this robustness test, through simulation with increasing time delay, the margin of the adaptive controller was found to be nearly identical to that of the nominal controller. This means that the conventional adaptive controller solves Problem 1 with respect to this margin. However, under ROE failure, its performance is degraded when tested with time delay. Figure 8 shows the behavior observed with a

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**Table 2 Nominal controller margins with ROE failure**

<table>
<thead>
<tr>
<th>Loop</th>
<th>Gain margin, dB</th>
<th>Phase margin, deg</th>
<th>TD margin, s</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOE</td>
<td>4.0</td>
<td>Inf</td>
<td>Inf</td>
</tr>
<tr>
<td>LME</td>
<td>5.5</td>
<td>20.0</td>
<td>0.13</td>
</tr>
<tr>
<td>LIE</td>
<td>-3.4</td>
<td>-24.7</td>
<td>0.16</td>
</tr>
<tr>
<td>RIE</td>
<td>-1.3</td>
<td>21.8</td>
<td>0.11</td>
</tr>
<tr>
<td>RMEa</td>
<td>2.3</td>
<td>8.4</td>
<td>0.05</td>
</tr>
<tr>
<td>ROE</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>TVC</td>
<td>-17.0</td>
<td>20.5</td>
<td>0.12</td>
</tr>
</tbody>
</table>

*Critical control loop*
Fig. 8 Performance of the adaptive controller under ROE failure with \( y = 200 \) and \( k_e = 0 \) for an RME time delay of 0.20 s.

Fig. 9 Performance of the adaptive controller under ROE failure with \( y = 500 \) and \( k_e = 100 \) for an RME time delay of 0.20 s.

Fig. 10 Performance of the adaptive controller under ROE failure with \( y = 200 \) and \( k_e = 200 \) for an RME time delay of 0.20 s.

Fig. 11 Comparison of \( J(t) \) without and with ALR for \( y = 500 \) and \( k_e = 200 \) under ROE failure and RME time delay of 0.20 s.

Fig. 12 Feedback signals with sensor noise.

Fig. 13 Performance of the adaptive controller with sensor noise for \( y = 500 \) and \( k_e = 200 \).
time delay in the RME channel of 0.20 s. Figure 9 shows the improvement obtained when the ALR modification was employed using $\Gamma = 500$, $k_w = 100$. Figure 10 shows that performance improves when $k_w$ is increased to 200. This illustrates that Problem 2 is solved for the case of ROE failure.

Several properties have been observed. As $k_w$ is increased, it is necessary to ensure that the step size used when integrating the adaptive law is sufficiently small. However, it is not necessary to sample the sensors or update the control at a higher rate. Also, with the introduction of $k_w > 0$, it is possible to increase the adaptation gain $\gamma$, thereby achieving more rapid adaptation. Figure 11 shows a comparison of the time history $J(t)$ in Eq. (10) without $(k_w = 0)$ and with $(k_w = 200)$ ALR modification.

Figures 12–15 illustrate results obtained when the adaptive controller with ALR modification was tested under the effect of sensor noise. These results are for the same failure condition and time delay as well as for the same parameter settings as that described for Fig. 10. Band-limited white noise was added to each element of $x_r(t)$, as shown in Fig. 12. The effect on the resulting trajectory variables for $k_w = 200$ is shown in Fig. 13. The response is very similar to the response shown in Fig. 10 that was obtained without sensor noise. Figure 14 shows the controller responses obtained with $k_w = 200$, and Fig. 15 shows the controller responses obtained with $k_w = 1000$. These figures validate the claim that increasing the ALR gain does not amplify the effect of sensor noise in the control.

### VI. Conclusions

A novel approach to robust synthesis of adaptive control systems has been presented. The underlying principle is to modify an existing weight adaptation law so that the columns of the weight matrix are rapidly attracted to and maintained in the neighborhood of a null space that preserves the robustness properties of the reference model. The modification term is easy to implement and can be used in a complementary way with many other approaches to adaptive control. The proof of stability relies on arguments taken from singular perturbation theory. Furthermore, it is shown that increasing the gain on the modification term does not adversely affect the control response to sensor noise. Example results indicate that this approach shows significant promise for flight control applications.

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### References


