CONTROL OF MULTIAGENT SYSTEMS UNDER PERSISTENT DISTURBANCES

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Multiagent Systems II
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1 Preliminaries
   - Multiagent Systems
   - Consensus and Formation Problems
   - Persistent Disturbance Issue

2 Control under Persistent Disturbances
   - Proposed Local Control Architecture

3 Stability and Convergence
   - Semistability
   - Null space convergence
   - Convergence to a Constant Point
   - Illustrative Examples

4 Generalizations
   - Time-Varying Disturbances
   - Illustrative Examples

5 Concluding Remarks
Outline

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5 Concluding Remarks
Agents able to sense, communicate, and process information.

Physical, biological, engineering, social, and economic systems.
Consensus Problem

Consensus

- Refers to agents coming to a global agreement on a state value

Graph Theoretic Modeling

- \( \dot{x}_i(t) = u_i(t), \quad u_i(t) \in \mathbb{R}^m, \quad i = 1, \ldots, n \)
- \( u_i(t) = -\sum_{j \sim i} (x_i(t) - x_j(t)) \)

\[
\Rightarrow \quad \dot{x}(t) = -\mathcal{L}(G) \otimes I_m x(t), \quad \mathcal{L}(G) \triangleq \mathcal{D}(G) - \mathcal{A}(G)
\]

where \( x(t) = [x_1^T(t), \ldots, x_n^T(t)]^T \)

- Connected (\( \leftrightarrow \)) graph and \( m = 1 \)

\[
\Rightarrow \quad \lim_{t \to \infty} x(t) = 1_n (1_n^T 1_n)^{-1} 1_n^T x_0 = \left(1_n 1_n^T / n\right) x_0
\]
Consensus Problem

Consensus

- Refers to agents coming to agreement on a state value

Graph Theoretic Modeling

\[ \dot{x}_i(t) = u_i(t), \quad u_i(t) = -\sum_{i \sim j}(x_i(t) - x_j(t)) \]

\[ \Rightarrow \dot{x}(t) = -L(G) \otimes \mathbf{1} x(t) \]

where \( x(t) = [x_1^T(t), \ldots, x_n^T(t)]^T \)

- Connected (\( \leftrightarrow \)) graph

\[ \Rightarrow \lim_{t \to \infty} x(t) = 1_n(1_n^T 1_n)^{-1} 1_n^T x_0 = (1_n 1_n^T / n) x_0 \]
Formation Problem

Formation

- Refers to agents moving toward a desired geometric shape

Graph Theoretic Modeling

- Define $\tau_i$ as the displacement of $x_i$ from the target location $\xi_i$
- Using transformation $\tau_i(t) = x_i(t) - \xi_i$ in $\dot{x}(t) = -\mathcal{L}(\mathcal{G})x(t)$

$$\Rightarrow \dot{x}(t) = -\mathcal{L}(\mathcal{G})x(t) - \mathcal{L}(\mathcal{G})\zeta$$

where $\zeta = [\zeta_1, \cdots, \zeta_n]^T$, or, locally

$$\Rightarrow \dot{x}_i(t) = -\sum_{j \sim i}(x_i(t) - x_j(t)) - (\zeta_i - \zeta_j)$$
Consensus and formation algorithms are not robust to disturbances

Need for controllers with integral action (at least second order)

Need to preserve the local nature of each agent's controller
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Control under Persistent Disturbances

Proposed Local Control Architecture

Problem Formulation

- System of \( n \) agents locally communicating subject to disturbances
  \[
  \dot{x}(t) = u(t) + w, \quad w = [w_1^T, \ldots, w_n^T]^T
  \]
- Reach global agreement on a state value

Proposed Controller

- \( u(t) = u_s(t) + u_a(t) \), \( u_s(t) = -\mathcal{L}(G)x(t) \), \( u_a(t) = -\hat{w}(t) \)
- \( \dot{\hat{w}}(t) = KQ(G)(x(t) - \hat{x}(t)) \)
  \[
  K = \text{diag}(k) > 0, \quad k = [k_1, \ldots, k_n]^T \quad \text{(learning gain)}
  \]
  \[
  Q(G) = I_n - S(G)(I_n + A(G))
  \]
  \[
  S(G) = \text{diag}(s) > 0, \quad s = [(N_1 + 1)^{-1}, \ldots, (N_n + 1)^{-1}]^T
  \]
  \[
  N_i, i = 1, \ldots, n, \text{ is the number of agent } i's \text{ neighbors}
  \]
- \( \dot{x}(t) = -\mathcal{L}(G)\hat{x}(t) + M(x(t) - \hat{x}(t)) \) \quad (\ast)
  \[
  M = mI_n > 0 \quad \text{(predicted state gain)}
  \]
Controller of Agent $i$

\[ u(t) = -\mathcal{L}(\mathcal{G})x(t) - KQ(\mathcal{G}) \int_{0}^{\infty} (x(t) - \hat{x}(t)) \, dt \]

- Standard control solves the consensus problem if $w = 0_n$
- $x(t) - \hat{x}(t)$ serves as an error signal
- Additional control $u_a(t)$ has an integral action
  - Readily minimizes this error multiplied from left by $KQ(\mathcal{G})$

**Local Projection Operator**

$Q(\mathcal{G})$ is local version of $P_{\mathcal{L}(\mathcal{G})} \triangleq \mathcal{L}(\mathcal{G})(\mathcal{L}(\mathcal{G})^T \mathcal{L}(\mathcal{G}))+\mathcal{L}(\mathcal{G})^T = I_n - (1_n 1_n^T \, / \, n)$

**Structure of Local Controllers**

- $u_i(t) = -\sum_{i \sim j} (x_i(t) - x_j(t)) - \hat{w}_i(t)$
- $\dot{\hat{w}}_i(t) = -[KS(\mathcal{G})]_{ii} \sum_{i \sim j} (\tilde{x}_i(t) - \tilde{x}_j(t))$, $\tilde{x}_i(t) \triangleq x_i(t) - \hat{x}_i(t)$
- $\dot{\hat{x}}_i(t) = -\sum_{i \sim j} (\hat{x}_i(t) - \hat{x}_j(t)) + m(x_i(t) - \hat{x}_i(t))$
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Semistability

- \( \tilde{x}(t) \triangleq x(t) - \hat{x}(t) \) \( \Rightarrow \) \( \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) - \tilde{w}(t) \), \( \tilde{A} \triangleq -\mathcal{L}(\mathcal{G}) - M \)
- \( \tilde{w}(t) \triangleq \hat{w}(t) - w \) \( \Rightarrow \) \( \dot{\tilde{w}}(t) = KQ(\mathcal{G})\tilde{x}(t) \)
- \( e(t) \triangleq [\tilde{x}^T(t), \tilde{w}^T(t)]^T \) \( \Rightarrow \) \( \dot{e}(t) = \tilde{A}_0 e(t) \), \( \tilde{A}_0 \triangleq \begin{bmatrix} \tilde{A} & -I_n \\ KQ(\mathcal{G}) & 0_{n \times n} \end{bmatrix} \)

Supporting Lemmas

- \( \tilde{A} = -\mathcal{L}(\mathcal{G}) - M \) is asymptotically stable
- \( KQ(\mathcal{G}) \) has \( n - 1 \) positive eigenvalues and a zero eigenvalue
- Quadratic matrix polynomial \( Z(\lambda) = A\lambda^2 + B\lambda + C \) with nonsingular \( A \in \mathbb{IR}^{n \times n} \), positive-definite \( B \in \mathbb{IR}^{n \times n} \), and \( C \in \mathbb{IR}^{n \times n} \) has
  - \( \pi_-(Z) = \pi_+(A) + \pi_+(C) \) and \( \pi_0(Z) = \pi_0(C) \)

Theorem

- The solution \( e(t) \) is Lyapunov stable for all \( e_0 \in \mathbb{IR}^{2n} \) and \( t \geq 0 \), and \( e(t) \to \epsilon [1_n^T, -m1_n^T]^T \) as \( t \to \infty \), where \( \epsilon \) is a constant in \( \mathbb{IR} \)
- \( Z(\lambda) = \lambda^2I_n + \lambda(-\tilde{A}) + KQ(\mathcal{G}) \)
Proposition

The proposed controller produces consensus, that is,

\[
\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \frac{1}{n} \left[ \frac{1}{n}^T x_0 - \frac{1}{n}^T \int_0^t \tilde{w}(\sigma) d\sigma \right]
\]

- Projections \( P_{\mathcal{L}(G)} \triangleq \mathcal{L}(G)(\mathcal{L}(G)^T \mathcal{L}(G))^+ \mathcal{L}(G)^T \) and \( P_{\perp \mathcal{L}(G)} \triangleq I_n - P_{\mathcal{L}(G)} \)

\[
\Rightarrow \lim_{t \to \infty} x(t) = \lim_{t \to \infty} P_{\mathcal{L}(G)} x(t) + \lim_{t \to \infty} P_{\perp \mathcal{L}(G)} x(t)
\]

\[
= \lim_{t \to \infty} x_1(t) + \lim_{t \to \infty} x_2(t)
\]

- \( \lim_{t \to \infty} x_1(t) = 0_n \)

- \( \lim_{t \to \infty} x_2(t) = \lim_{t \to \infty} P_{\perp \mathcal{L}(G)} x(t) = \lim_{t \to \infty} \frac{1}{n}^{1T}_n x(t) \)

\[
= \lim_{t \to \infty} \left( \frac{1}{n}^{1T}_n e^{-\mathcal{L}(G)t} x_0 - \frac{1}{n}^{1T}_n \int_0^t e^{-\mathcal{L}(G)(t-\sigma)} \tilde{w}(\sigma) d\sigma \right)
\]

\[
= \lim_{t \to \infty} \left( \frac{1}{n}^{1T}_n x_0 - \int_0^t \frac{1}{n}^{1T}_n \tilde{w}(\sigma) d\sigma \right)
\]
Boundedness of the Control Signal

**Proposition**

The control signal \( u(t) = -\mathcal{L}(G)x(t) - KQ(G)\int_0^\infty (x(t) - \hat{x}(t))\,dt \) satisfies \( \|u(t)\|_2 \leq u^* \) for all \( t \geq 0 \).

\[ \dot{x}(t) = Ax(t) + d(t), \quad A \triangleq -\mathcal{L}(G), \quad d(t) \triangleq -\hat{w}(t) \]

- Defining \( P_\Sigma \triangleq [p_1, \ldots, p_{n-1}, \epsilon_1 n], \quad p_i \in \mathbb{R}^n \)

\[ \Rightarrow A = P_\Sigma \Sigma P_\Sigma^T, \quad \Sigma \triangleq \begin{bmatrix} \Sigma_0 & 0_{n-1} \\ 0_{n-1}^T & 0 \end{bmatrix} \]

- Applying \( z(t) = P_\Sigma^T x(t) \) with \( P_\Sigma^T d(t) = [d_1^T(t), \ d_2(t)]^T \)

\[ \Rightarrow \dot{z}_1(t) = \Sigma_0 z_1(t) + d_1(t) \Rightarrow \text{Bounded (} \Sigma \text{ Hurwitz)} \]

\[ \Rightarrow \dot{z}_2(t) = d_2(t) \text{ (do not care)} \]

\[ u(t) = Ax(t) - \hat{w}(t) = P_\Sigma \begin{bmatrix} \Sigma_0 z_1(t) \\ 0 \end{bmatrix} - \hat{w}(t) \Rightarrow \text{Bounded} \]
Extensions to Formation Problem

**Proposed Controller**

\[ u(t) = u_s(t) + u_a(t) + u_f(t) \]

- \[ u_s(t) = -\mathcal{L}(\mathcal{G})x(t) \]
- \[ u_a(t) = -\hat{w}(t) \]
  - \[ \dot{\hat{w}}(t) = K Q(\mathcal{G})(x(t) - \hat{x}(t)) \]
  - \[ \dot{\hat{x}}(t) = -\mathcal{L}(\mathcal{G})\hat{x}(t) + M(x(t) - \hat{x}(t)) + u_f(t) \]
- \[ u_f(t) = -\mathcal{L}(\mathcal{G})\zeta \]

\[ e(t) \triangleq [\hat{x}^T(t), \hat{w}^T(t)]^T \Rightarrow \dot{e}(t) = \tilde{A}_0 e(t), \quad \tilde{A}_0 \triangleq \begin{bmatrix} \tilde{A} & -I_n \\ K Q(\mathcal{G}) & 0_{n \times n} \end{bmatrix} \]

- **Same** error dynamics (**stability holds**)!
- **Convergence** is a direct consequence

\[ \lim_{t \to \infty} x(t) = \zeta + \lim_{t \to \infty} \frac{1}{n} \left[ 1_n^T x_0 - 1_n^T \int_0^t \hat{w}(\sigma) d\sigma \right] \]
Illustrative Example 1: $u_s \neq 0$ and $u_a \equiv 0$

- Cycle graph with 6 agents subject to constant disturbances
Illustrative Example 1: $u_s \neq 0$ and $u_a \neq 0$

- Cycle graph with 6 agents subject to constant disturbances
Convergence to a Constant Point

For consensus on a constant point in space

\[ \lim_{t \to \infty} x(t) = \lim_{t \to \infty} \frac{1}{n} \left[ 1^T_n x_0 - 1^T_n \int_0^t \tilde{w}(\sigma) d\sigma \right] \]

\[ \Rightarrow \lim_{t \to \infty} \hat{w}(t) = 0_n \]

Modifying Integral Action

\[ \hat{w}(t) = K(Q(G) + qI_n)(x(t) - \hat{x}(t)), \quad q \in \mathbb{IR}_+ \]

\[ \hat{w}_i(t) = -[KS(G)]_{ii} \sum_{i \sim j} (\tilde{x}_i(t) - \tilde{x}_j(t)) + q[K]_{ii} \tilde{x}_i(t) \]

By \( e(t) \triangleq [\tilde{x}^T(t), \tilde{w}^T(t)]^T \) we have

\[ \dot{e}(t) = \begin{bmatrix} \tilde{A} & -I_n \\ K(Q(G) + qI_n) & 0_{n \times n} \end{bmatrix} e(t) \]

\[ \mathcal{Z}(\lambda) = \lambda^2 I_n + \lambda(-\tilde{A}) + (KQ(G) + qI_n) \Rightarrow \lim_{t \to \infty} e(t) = 0 \]
Illustrative Example 1 (Revisited)

- Cycle graph with 6 agents subject to constant disturbances
Illustrative Example 2: \( u_s \neq 0 \) and \( u_a \equiv 0 \)

- Cycle graph with 6 agents subject to constant disturbances
- Formation objective: \( \zeta = [0.0, 0.2, 0.4, 0.6, 0.8, 1.0]^T \)
Illustrative Example 2: $u_s \neq 0$ and $u_a \neq 0$

- Cycle graph with 6 agents subject to constant disturbances
- Formation objective: $\zeta = [0.0, 0.2, 0.4, 0.6, 0.8, 1.0]^T$
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**Problem Formulation**

- System of \( n \) agents locally communicating subject to disturbances
  \[ \dot{x}(t) = u(t) + w(t), \quad \|w(t)\|_2 \leq w^*, \quad \|\dot{w}(t)\|_2 \leq \dot{w}^* \]
- Reach approximate global agreement on a state value

By \( e(t) \triangleq [\tilde{x}^T(t), \tilde{w}^T(t)]^T \) we have

\[
\dot{e}(t) = \begin{bmatrix}
\tilde{A} & -I_n \\
KQ(G) & 0_{n \times n}
\end{bmatrix} e(t) + \begin{bmatrix}
0_n \\
-I_n
\end{bmatrix} \dot{w}(t)
\]

- Additional input \( \dot{w}(t) \) can lead to “bursting”
### Proposed Controller

- $u(t) = u_s(t) + u_a(t)$
  - $u_s(t) = -\mathcal{L}(G)x(t)$
  - $u_a(t) = -\hat{w}(t)$
    - $\hat{w}(t) = K\left( Q(G)(x(t) - \hat{x}(t)) - \kappa \hat{w} \right)$, $\kappa \in \mathbb{R}^+$
    - $\hat{x}(t) = -\mathcal{L}(G)\hat{x}(t) + M(x(t) - \hat{x}(t))$

- $V(\tilde{x}, \tilde{w}) = \tilde{x}^T \tilde{x} + \tilde{w}^T K^{-1} \tilde{w}$
  - $\dot{V}(\cdot) \leq -c_1 \|\tilde{x}\|_2^2 - c_2 \|\tilde{w}\|_2^2 + c_3$, $c_i > 0$, $i = 1, 2, 3$
  - Uniform ultimate boundedness

- Ultimate bound is small if $\lambda_{\text{min}}(K)$ large and/or $\kappa$ small
Illustrative Example 3: \( u_s \neq 0 \) and \( u_a \equiv 0 \)

- Cycle graph with 6 agents subject to time-varying disturbances
Illustrative Example 3: $u_s \neq 0$ and $u_a \neq 0$

- Cycle graph with 6 agents subject to time-varying disturbances
Illustrative Example 4: $u_s \neq 0$ and $u_a \neq 0$

- Cycle graph with 6 agents subject to time-varying disturbances
- Formation objective: $\zeta = [0.0, 0.2, 0.4, 0.6, 0.8, 1.0]^T$
Illustrative Example 5: 2D Case

- Cycle graph with 6 agents subject to time-varying disturbances
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Concluding Remarks

Conclusions

- **Consensus** and **formation** problems are investigated when the agents are **perturbed** by (unknown) persistent disturbances.
- Proposed controller(s) **suppresses** the effect of **constant** or **time-varying** disturbances.
- Realization only requires **local information**.

Ongoing Research

- Analysis of **design parameters’ effects** on stability margins.
- Extensions to **digraphs** and **dynamic graphs**.
- Extensions to **$n$-th order** multivehicle systems.
Thank You