\( K \)-modification based \( \mathcal{H}_2 \) Adaptive Control

Kilsoo Kim*, Tansel Yucelen**, and Anthony J. Calise***

*Georgia Institute of Technology, Atlanta, GA, 30332

Two novel adaptive control laws are presented. The first is based on the use of a tunable stiffness term that provides a frequency dependent filtering effect, smoother transient responses, and time delay robustness in the adaptive system. The second is an \( \mathcal{H}_2 \) approach which extends a recently developed Kalman filter based approach to adaptive control law modification, by providing an optimal time varying adaptation gain. The proposed \( \mathcal{H}_2 \) model reference adaptive control law with \( K \)-modification combines the advantages in each method. It can also be used with the well known \( \sigma \)- and \( \varepsilon \)-modification terms.

I. Introduction

There have been numerous research efforts in the adaptive control literature to improve model reference adaptive control laws. Modification terms related to adding damping are commonly introduced in the weight update law. So-called \( \sigma \)-modification\(^1\) adds a pure damping term, while \( \varepsilon \)-modification\(^2\) adds a variable damping term that depends on the magnitude of the error signal. In general, these modification terms are added to ensure that the adapted weights remain bounded. A projection operator\(^3\) has also been used to bound the weights. Background learning\(^4,5\) uses current and past data concurrently in the adaptation process to improve adaptive system performance by allowing the adaptation law to continually train in the background based on the past data while still being responsive to dynamic changes based on the current data. \( Q \)-modification\(^6\) was intended to improve adaptation performance by using a moving window of the integrated system uncertainty. An optimal control theory based modification term\(^7\) was also developed in order to improve adaptation in the presence of large adaptive gain. More recently, an adaptive loop recovery (ALR) approach\(^8\) has been introduced as a modification term with the goal of recovering the loop transfer properties of a reference model. A Kalman filter (KF) based approach\(^9,10\) has been suggested as a modification method that more effectively achieves the objectives of all the previously mentioned modification methods through better conditioning in terms of the magnitude of the adaptation gain required for a given level of tracking performance and the associated level of control activity.

\( K \)-modification\(^11\) introduces a stiffness term to the aforementioned methods of modification. When combined with well known \( \sigma \)- and \( \varepsilon \)-modification it provides of means for manipulating the natural frequency and damping ratio in the error transients. \( K \)-modification also has a natural robustness to time delay. The KF optimization approach has been extended to \( \mathcal{H}_2 \) adaptive control (\( \mathcal{H}_2 \)-AC)\(^12\) by generalizing the linear constraint assumption\(^9,10\) and enforcing this constraint on the weights by employing \( \mathcal{H}_2 \) optimization. This generalization allows one to replace a gradient based adaptive control law with its \( \mathcal{H}_2 \) form, rather than simply replace the modification term as was done in the KF approach. The system response of \( \mathcal{H}_2 \)-AC is shown to have smaller tracking errors without requiring high adaptation gains when compared with the gradient based adaptive control form.

A model reference adaptive control law, referred to as \( \mathcal{H}_2^2 K \)-AC, is proposed in this paper that combines \( \mathcal{H}_2 \)-AC with \( \sigma \)- and \( K \)-modification methods to combine the benefits provided by each of these individual methods. It is shown that the \( \mathcal{H}_2^2 K \)-AC combination is preferable to the combination of \( \mathcal{H}_2 \)-AC with \( K \)- and \( \varepsilon \)-modification in view of prescribed damping and stiffness.

*Graduate Research Assistant, Student Member AIAA, School of Aerospace Engineering, kilsoo.kim@gatech.edu.
**Graduate Research Assistant, Student Member AIAA, School of Aerospace Engineering, tansel@gatech.edu.
***Professor, Fellow AIAA, School of Aerospace Engineering, anthony.calise@aerospace.gatech.edu.
The paper is organized as follows. Section II provides a brief background on model reference adaptive control needed for the remaining sections. In section III, $\mathcal{K}$-modification is defined and its characteristics are interpreted in the context of model reference adaptive control. In section IV, $\mathcal{H}_2$-AC is combined with $\sigma$- and $\mathcal{K}$-modification. In addition a stability analysis is provided. In section V, a numerical example illustrates the properties of $\mathcal{K}$-modification and $\mathcal{H}_2^2\mathcal{K}$-AC through a wing-rock simulation. Conclusion and future directions are given in Section VI.

II. Preliminaries in Model Reference Adaptive Control

Consider the uncertain nonlinear system

$$\dot{x}(t) = Ax(t) + B[u(t) + \Delta(x(t))]$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ are known matrices, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $\Delta(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is the unknown uncertainty, and system (1) is stabilizable. The baseline control for the system (1) is given by

$$u_n(t) = -K_x x(t) + K_r r(t)$$

where $r(t) \in \mathbb{R}^r$ is the bounded reference command and $K_x \in \mathbb{R}^{m \times n}$ and $K_r \in \mathbb{R}^{m \times r}$ are given state and input gain matrices. It is assumed that the matched uncertainty in (1) can be linearly parameterized in the form of

$$\Delta(x(t)) = W^T(\beta(x(t)) + \varepsilon(x(t)))$$

where $W \in \mathbb{R}^{q \times m}$ is the unknown constant weight matrix and $\beta(\cdot) : \mathbb{R}^n \to \mathbb{R}^q$ is the bounded basis function vector of the form $\beta(x(t)) = [\beta_1(x(t)), \beta_2(x(t)), ..., \beta_{q-1}(x(t))]^T \in \mathbb{R}^q$, where $\varepsilon(\cdot) : \mathbb{R}^n \to \mathbb{R}^m$ is the residual error satisfying $|\varepsilon(x(t))| < \varepsilon$ on a sufficiently large bounded set in the state space.

With the system and baseline controller matrices given, a reference model can be constructed in the form

$$\dot{x}_m(t) = A_m x_m(t) + B_m r(t)$$

where $x_m(t) \in \mathbb{R}^n$ is the model state vector, $A_m = A - BK_x$, $B_m = BK_r$, and $A_m$ is Hurwitz by design. The objective is to augment the baseline control law in (2) so that $x(t)$ asymptotically tracks $x_m(t)$. The augmented control has the form

$$u(t) = u_n(t) - u_{ad}(t)$$

where

$$u_{ad}(t) = \dot{W}(t)^T \beta(x(t))$$

satisfies the adaptive law\(^1\)\(^2\)

$$\dot{W}(t) = \gamma \beta(x(t)) e(t)^T PB - \dot{W}_m(t)$$

where $\gamma$ is the positive adaptive learning rate, $e(t) = x(t) - x_m(t)$, $\dot{W}_m(t)$ is a modification term, i.e. $\dot{W}_m(t) = \sigma \dot{W}(t)$ for $\sigma$-modification\(^1\) and $\dot{W}_m(t) = \lambda \|e(t)\|^2 \dot{W}(t)$ for $e$-modification\(^2\) with the positive damping learning rates of $\sigma$ and $\lambda$, and $P$ is a positive-definite solution of the Lyapunov equation for any $Q > 0$

$$0 = A_m^T P + PA_m + Q$$

Figure 1 summarizes the conventional structure of the model reference adaptive controller as described above. The weight estimation process in the adaptation box will be explained in detail in the following two sections (III and IV).
III. \( K \)-modification

The motivation behind \( K \)-modification is the desire to achieve a prescribed natural frequency and damping ratio for the error transients. In this section, \( K \)-modification is defined by introducing a stiffness term in the weight update law. The second order characteristics of the update law is studied with respect to damping ratio and natural frequency with a combination of \( \sigma \)- and \( e \)-modification terms. The weight feedback effect coming from adding a stiffness term in the weight update law is described in the state space representation. The unknown weight in (3) is estimated using the modified adaptive law

\[
\dot{W}(t) = \gamma \beta(x(t)) e^T(t) PB - \dot{W}_m(t) - k \int_{t-T}^{t} \dot{W}(s) ds
\]

where \( k \) is the positive stiffness learning rate and \( T > 0 \) appears by adding an integral form of stiffness to the first order weight update law.

First consider the single input case \( (m = 1) \) to simply illustrate the characteristics of the \( K \)-modified update law in (9). Differentiate the update law in (9) with respect to time and apply the following error dynamics

\[
\dot{e}(t) = A_m e(t) - BW(t) \beta(x(t)) + Be(x(t))
\]

where \( \dot{W}(t) = \dot{W}(t) - W \). Then the update law in (9) can be written in the following second order form

\[
\dot{W}(t) + C_w \dot{W}(t) + K_w W(t) = F_w
\]

where the damping \( C_w \), stiffness \( K_w \), and forcing \( F_w \) terms are expressed below. When the \( \sigma \)-modification term is used with \( K \)-modification,

\[
C_w = \sigma
\]

\[
K_w = \gamma \left[ \beta_c(x(t)) Be^T(t) + \beta(x(t)) B^T \right] PB \beta^T(x(t)) + k
\]

\[
F_w = \gamma \left[ \beta_c(x(t)) \left( A_m x(t) + B_m r(t) + B \Delta(x(t)) \right) e^T(t) + \beta(x(t)) \left( A_m e(t) + B \Delta(x(t)) \right)^T \right] PB
\]

When the \( e \)-modification is used with \( K \)-modification,

\[
C_w = \lambda \| e(t) \|
\]

\[
K_w = \gamma \left[ \beta_c(x(t)) Be^T(t) + \beta(x(t)) B^T \right] PB \beta^T(x(t)) + \lambda \frac{e^T(t) \dot{e}(t)}{\| e(t) \|} + k
\]

\[
F_w = \gamma \left[ \beta_c(x(t)) \left( A_m x(t) + B_m r(t) + B \Delta(x(t)) \right) e^T(t) + \beta(x(t)) \left( A_m e(t) + B \Delta(x(t)) \right)^T \right] PB
\]

where \( \dot{W}(t - T) \) is the \( T \) time delayed value of \( \dot{W}(t) \).
Remark 1: The characteristics of the second order form of the estimated update law in (11) are analogous to those of a mechanical system. The effects of $K$-modification on the update law appear only in the stiffness term $K_w$ and the forcing term $F_w$. The stiffness term is affine in $k$ and forcing term can be viewed as an input that is filtered by the adaptive law. The effects of $\sigma$- and $e$- modification on the update law appear in the damping term $C_w$ in both cases. Note that $C_w$ only depends on $\sigma$ and that $K_w$ is independent of $\sigma$ in the case of $\sigma$-modification, whereas $e$-modification affects both $C_w$, in a manner that depends on $\|e(t)\|$, and $K_w$ in a more complicated manner.

For simplicity, in the case $m = 1$, the filtering effect of the second order weight update law can be characterized in terms of an equivalent damping ratio and natural frequency:

For the case of $\sigma$- and $K$-modification,

$$\zeta(t) = \frac{C_w(t)}{2\sqrt{K_w(t)}} = \frac{\sigma}{2\sqrt{\gamma \left[ \beta_x(x(t))Be^T(t) + \beta(x(t))B^T \right] PB\beta^T(x(t)) + k}}$$

(14)

$$\omega_n(t) = \sqrt{K_w(t)} = \sqrt{\gamma \left[ \beta_x(x(t))Be^T(t) + \beta(x(t))B^T \right] PB\beta^T(x(t)) + k}$$

(15)

For the case of $e$- and $K$-modification,

$$\zeta(t) = \frac{C_w(t)}{2\sqrt{K_w(t)}} = \frac{\lambda\|e(t)\|}{2\sqrt{\gamma \left[ \beta_x(x(t))Be^T(t) + \beta(x(t))B^T \right] PB\beta^T(x(t)) + \lambda\frac{e^T(t)e(t)}{\|e(t)\|^2} + k}}$$

(16)

$$\omega_n(t) = \sqrt{K_w(t)} = \sqrt{\gamma \left[ \beta_x(x(t))Be^T(t) + \beta(x(t))B^T \right] PB\beta^T(x(t)) + \lambda\frac{e^T(t)e(t)}{\|e(t)\|^2} + k}$$

(17)

Remark 2: It is clear from (14)-(17) that the product $\zeta(t)\omega_n(t)$ is proportional to $\sigma$ in the case of $\sigma$-modification and to $\lambda\|e(t)\|$ in the case of $e$-modification. In an underdamped second order system, this product determines the settling time. The above suggests that for a given $\sigma$, one should select $k$ so that the resulting damping ratio is reasonable for a given magnitude of error. It is interesting to note that in the case of $\sigma$- modification, the damping ratio is independent of $\|e(t)\|$, whereas (16) suggests that $e$- modification has the effect of causing the damping ratio to approach zero as $\|e(t)\|$ approaches zero.

The second order weight update law (11) is expressed in state-space form to further examine the properties of weight update law,

$$\dot{\Sigma}(t) = A_{\Sigma}(t)\Sigma(t) + B_{\Sigma}F_w(t)$$

$$\Sigma_y(t) = C_{\Sigma}\Sigma(t)$$

(18)

where $\Sigma(t) = [\dot{W}^T(t), \dot{W}^T(t)]^T \in \mathbb{R}^{2q}$ is the state, $\Sigma_y(t) = \dot{W}(t) \in \mathbb{R}^q$ is the output, $F_w(t) \in \mathbb{R}^q$ is the input, and $A_{\Sigma}(t) \in \mathbb{R}^{2q \times 2q}$, $B_{\Sigma} \in \mathbb{R}^{2q \times q}$, $C_{\Sigma} \in \mathbb{R}^{q \times 2q}$ are expressed as

$$A_{\Sigma}(t) = \begin{bmatrix} 0_{q \times q} & I_{q \times q} \\ -K_w(t) & -C_w \end{bmatrix}, \quad B_{\Sigma} = [0_{q \times q}, I_{q \times q}]^T, \quad C_{\Sigma} = [I_{q \times q}, 0_{q \times q}]$$

(19)

In comparison to $e$- and $\sigma$-modification, $K$-modification can be viewed as containing a feedback of the delayed weight estimates, which is a consequence of introducing the stiffness term in the update law. This is illustrated in Figure 2, where $F_{\mu}(t)$ is the value of $F_w(t)$ for $k = 0$. $F_{\mu}(t)$ may have oscillations, which can be reduced by the feedback of the $T$ time delayed estimated weight $\hat{W}(t - T)$. When $T$ is selected around the value of the time delay applied in the system, it is shown to have good performance in the wing-rock example to be presented later. This suggests that $K$-modification may provide an opportunity to treat time delay by setting $T$ equal to an estimate of the actual time delay.
In the following theorem we employ the notation $\text{Proj}(\cdot, \cdot)$ to denote the projection operator.\(^3\)

**Theorem 1:** Consider the uncertain nonlinear system given by (1) with $u(t)$ given by (5) and reference model given by (4). Then the closed loop error system defined by (10) and (11) with weight update law (22) can be written as

$$
\dot{\hat{W}}(t) = \text{Proj}\left(\hat{W}(t), \gamma\beta(x(t))e^T(t)PB - X\right) + X
$$

(20)

where $X \triangleq \dot{\hat{W}}_m(t) + k \int_{t-T}^t \dot{\hat{W}}(s)ds$, is uniformly ultimately bounded (UUB).

**Proof:** Define $\mu(t) = [e(t)T \text{vec}(\hat{W}(t)T)]^T$ and $B_r = \mu(t) : |\mu(t)| \leq r$ such that $B_r \subset D$ for a sufficiently large $D$. Consider the Lyapunov function candidate

$$
V(e(t), \hat{W}(t)) = \frac{1}{2} e^T(t) Pe(t) + \frac{1}{2\gamma} \text{tr}[\hat{W}^T(t)\hat{W}(t)]
$$

(21)

To show ultimate boundedness, the proposed weight update law in (20) is used to update the NN weights to force the adaptive parameters to evolve in a prescribed compact region. The derivative of (21) with respect to time is written as

$$
\dot{V} = e^T(t)P\left[A_m e(t) - BW^T(t)\beta(x(t)) + B\varepsilon(x(t))\right] + \frac{1}{\gamma} \text{tr}\left[\hat{W}^T(t)\dot{\hat{W}}(t)\right]
$$

$$
= -\frac{1}{2} e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) - e^T(t)PB\hat{W}^T(t)\beta(x(t))
$$

(22)

$$
+ \frac{1}{\gamma} \text{tr}\left[\hat{W}^T(t)\text{Proj}\left(\hat{W}(t), \gamma\beta(x(t))e^T(t)PB - X\right) + \hat{W}^T(t)X\right]
$$

Using the following properties of the trace operator

$$
e^T(t)PB\hat{W}^T(t)\beta(x(t)) = \text{tr}[\hat{W}^T(t)\beta(x(t))e^T(t)PB]
$$

(23)

(22) can be written as

$$
\dot{V} = -\frac{1}{2} e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) - \text{tr}\left[\hat{W}^T(t)\beta(x(t))e^T(t)PB\right]
$$

$$
+ \frac{1}{\gamma} \text{tr}\left[\hat{W}^T(t)\text{Proj}\left(\hat{W}(t), \gamma\beta(x(t))e^T(t)PB - X\right) + \hat{W}^T(t)X\right]
$$

$$
= -\frac{1}{2} e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t))
$$

$$
+ \frac{1}{\gamma} \text{tr}\left[\hat{W}^T(t)\text{Proj}\left(\hat{W}(t), \gamma\beta(x(t))e^T(t)PB - X\right) - \hat{W}^T(t)\left(\gamma\beta(x(t))e^T(t)PB - X\right)\right]
$$

(24)

Letting $Y(t) = \gamma\beta(x(t))e^T(t)PB - X$, then (24) can be expressed as

$$
\dot{V} = -\frac{1}{2} e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) + \frac{1}{\gamma} \text{tr}\left[\hat{W}^T(t)\{\text{Proj}(\hat{W}(t), Y(t)) - Y(t)\}\right]
$$

(25)
A property of the trace operator is that inequality
\[ \text{tr} \left[ (\hat{\Theta} - \Theta)^T \{ \text{Proj}(\hat{\Theta}, Y) - Y \} \right] \leq 0 \]  
(26)

Therefore (25) becomes
\[ \dot{V} \leq -\frac{1}{2} e^T(t) Q e(t) + e^T(t) P B \varepsilon(x(t)) \leq -\|e(t)\| \left( c_1 \|e(t)\| - c_2 \right) \]  
(27)

where \( c_1 = \lambda_{\text{min}}(Q)/2 \), \( c_2 = \|PB\|\varepsilon, \|\varepsilon(x(t))\| < \varepsilon, \varepsilon > 0 \).

Define
\[ \Omega_e = \left\{ e \in \mathbb{R}^n : \|e\| \leq \frac{c_2}{c_1} = \frac{2\|PB\|\varepsilon}{\lambda_{\text{min}}(Q)} \right\} \]  
(28)

Then the time derivative of \( V(e(t), \hat{W}(t)) \) is negative definite outside of \( \Omega_e \) in (28). If \( \mu(t(0)) \in B_r \) then the closed-loop system trajectories given by the error dynamics (10) and the \( K \)-modified estimated update law (20) are uniformly ultimately bounded. In other words the closed-loop error \( \mu(t) \) enters the compact set \( \Omega_e \) in finite time and will remain inside the set for all future times.

Figure 3 illustrates the sets employed in the proof of Theorem 1.

Figure 3. Geometric representation of sets in the Theorem 1

IV. \( \mathcal{H}_2 \) with \( K \)-modification

In this section, a model reference adaptive control law is obtained by combining an \( \mathcal{H}_2 \) optimization approach\(^{12} \) with \( \sigma \)- and \( K \)-modification,\(^{11} \) which is referred to as \( \mathcal{H}_2^*K \). This proposed \( \mathcal{H}_2^*K \) adaptive control law preserves the advantages offered in each method. It provides an optimal time varying adaptation gain leading to improved performance without tuning of the adaptation gain, a tunable damping and stiffness, and a natural robustness to time delay.

Assumption 1: The constraint on the ideal weight matrix in an adaptive control design has the linear form
\[ W^T(t)\phi_1(t, x(t), u(t)) = \phi_2(t, x(t), u(t)) \]  
(29)

where \( W(t) \in \mathbb{R}^{s \times m} \) is an unknown weight matrix, \( \phi_1(\cdot) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{s \times 1} \) is a given regressor, and \( \phi_2(\cdot) : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times 1} \) is also a given regressor.

The problem of estimating \( W(t) \) while enforcing the linear constraint in (29) can be viewed as a standard problem in estimation theory by defining the stochastic process
\[ \dot{W}(t) = D_{w1}q(t) \]
\[ y(t) = \phi_1(t, x(t), u(t))^T W(t) + D_{w2}q(t) \]

where \( q(t) \) is a zero-mean, Gaussian, white noise process, and \( D_{w1}^T D_{w2} \) is assumed to be zero. The estimated weight update law is then given by

\[ \dot{\hat{W}}(t) = -S(t)\phi_1(t, x(t), u(t)) R(t)^{-1} \left[ \phi_1(t, x(t), u(t))^T \hat{W}(t) - \phi_2(t, x(t), u(t))^T \right] \]

where \( R = D_{w2}D_{w1}^T > 0 \) and \( S(t) \) satisfies the differential Riccati equation given by

\[ \dot{S}(t) = -S(t)\phi_1(t, x(t), u(t)) R(t)^{-1} \phi_1(t, x(t), u(t))^T S(t) + Q \]

where \( Q = D_{w1}^T D_{w1} > 0 \). The stability proof of the \( H_2-AC \) law given by (31) and (32) depends on the constraint given by (29) and is thus problem dependent. Hence we need to limit our presentation by providing the \( H_2-AC \) forms of well known \( \sigma- \) and \( \epsilon \)-modification based adaptive control laws.

Let \( \phi_1(t, x(t), u(t)) \) and \( \phi_2(t, x(t), u(t)) \) be the following

\[ \phi_1(t, x(t), u(t)) = (\sqrt{\sigma} + \sqrt{k}) I \]
\[ \phi_2(t, x(t), u(t)) = \frac{1}{\sqrt{\sigma}} \beta(x(t)) e^T(t) PB - \frac{1}{\sqrt{k}} \int_{t-T}^t \dot{W}(s) ds \]

Then, from the constraint in (29) and the equations in (31)-(32), the \( \sigma- \) and \( \mathcal{K} \)-modification based \( H_2 \) adaptive control (\( H_2^\sigma \mathcal{K}-AC \)) law can be given by

\[ \dot{\hat{W}}(t) = S(t) R^{-1} \left[ k_\sigma \beta(x(t)) e^T(t) PB - \bar{\sigma} k \hat{W}(t) - \sigma_k \int_{t-T}^t \dot{W}(s) ds \right] \]
\[ \dot{S} = -S(t) R^{-1} S(t) \bar{\sigma} k + Q \]

where \( k_\sigma = 1 + \sqrt{k/\sigma} > 1 \), \( \sigma_k = 1 + \sqrt{\sigma/k} > 1 \), and \( \bar{\sigma} k = \left( \sqrt{\sigma} + \sqrt{k} \right)^2 \)

**Theorem 2:** The nonlinear uncertain dynamical system given by (1), with the control law given by (5) and with the projection operator combined \( H_2^\sigma \mathcal{K}-AC \)

\[ \dot{\hat{W}}(t) = \Gamma(t) \left\{ \text{Proj} \left( \hat{W}(t), \beta(x(t)) e^T(t) PB - X_h \right) + X_h \right\} \]
\[ \dot{S} = -S(t) R^{-1} S(t) \bar{\sigma} k + Q \]

where \( \Gamma = S(t) R^{-1} \) and \( X_h \triangleq \left( \bar{\sigma} k \hat{W}(t) + \sigma_k \int_{t-T}^t \dot{W}(s) ds \right) / k_\sigma \), is UUB.

**Proof:** Consider the Lyapunov function candidate

\[ \mathcal{V}(e(t), \hat{W}(t)) = \frac{1}{2} e^T(t) P e(t) + \frac{1}{2} \text{tr} \left[ \dot{W}^T(t) \Gamma^{-1}(t) \dot{W}(t) \right] \]

where \( \Gamma(t) = S(t) R^{-1} \) is positive definite and uniformly bounded \( H_2 \) adaptation gain. To show ultimate boundedness, the proposed weight update law in (35) is used to update the NN weights to force the adaptive parameters to evolve in a prescribed compact region. The time derivative of (36) is written as

\[ \dot{\mathcal{V}} = e^T(t) P \left[ A_m e(t) - B \dot{\hat{W}}^T(t) \beta(x(t)) + B \varepsilon(x(t)) \right] \]
\[ + \text{tr} \left[ \dot{W}^T(t) \Gamma(t) \dot{\hat{W}}(t) \right] + \frac{1}{2} \text{tr} \left[ \dot{W}^T(t) \frac{d}{dt} (\Gamma^{-1}(t)) \dot{W}(t) \right] \]

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Using the Lyapunov equation in (8) and the weight update law in (35), (37) can be written as

\[
\dot{V} = -\frac{1}{2}e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) - e^T(t)PB\dot{W}(t)\beta(x(t)) \\
+ \text{tr} \left[ \dot{W}^T(t)\text{Proj} \left( \dot{W}(t), \beta(x(t))e^T(t)PB - X_h \right) \right] + \dot{W}^T(t)X_h \\
+ \frac{1}{2}\text{tr} \left[ \dot{W}^T(t) \frac{d}{dt} \left( \Gamma^{-1}(t) \right) \dot{W}(t) \right]
\]  

(38)

Using the property of trace operator in (23), (38) can be expressed as

\[
\dot{V} = -\frac{1}{2}e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) \\
+ \text{tr} \left[ \dot{W}^T(t)\text{Proj} \left( \dot{W}(t), \beta(x(t))e^T(t)PB - X_h \right) \right] \\
+ \frac{1}{2}\text{tr} \left[ \dot{W}^T(t) \frac{d}{dt} \left( \Gamma^{-1}(t) \right) \dot{W}(t) \right]
\]  

(39)

Letting \( Y_h = \beta(x(t))e^T(t)PB - X_h \), (39) becomes

\[
\dot{V} = -\frac{1}{2}e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) \\
+ \text{tr} \left[ \dot{W}^T(t) \{ \text{Proj} \left( \dot{W}(t), Y(t) \right) \} \right] \\
+ \frac{1}{2}\text{tr} \left[ \dot{W}^T(t) \frac{d}{dt} \left( \Gamma^{-1}(t) \right) \dot{W}(t) \right]
\]  

(40)

Also using the inequality property of projection operator in (26) and \( \frac{d}{dt} (\Gamma^{-1}(t)) = -RS^{-2}(t)\dot{S}(t) \), we have

\[
\dot{V} \leq -\frac{1}{2}e^T(t)Qe(t) + e^T(t)PB\varepsilon(x(t)) + \frac{1}{2}\text{tr} \left[ \dot{W}^T(t) \left( \sigma_k I - M \right) \dot{W}(t) \right]
\]  

(41)

where \( M = (S(t)R^{-1})^{-2}QR^{-1} \). From (41) we have that

\[
\dot{V} \leq -d_1 \|e(t)\|^2 - d_2 \|\dot{W}(t)\|^2 + d_3
\]  

(42)

where \( d_1 = \lambda_{\min}(Q)/2, d_2 = \lambda_{\min}(M - \sigma_k I)/2, d_3 = \|PB\|\|\varepsilon(x(t))\| < \epsilon, \) and \( \epsilon > 0 \).

Define

\[
\Omega_d = \left\{ e(t) \in \mathbb{R}^n, W(t) \in \mathbb{R}^{p \times m} : d_1 \|e(t)\|^2 + d_2 \|\dot{W}(t)\|^2 \leq d_3 \right\}
\]  

(43)

Then, \( \dot{V}(e(t), W(t)) \) is negative definite outside of \( \Omega_d \) in (43). If \( \mu(t(0)) \in \mathcal{B}_r \) then the closed-loop system trajectories given by \( \mathcal{H}_2^2K \)-modified estimated update law (35) and error dynamics (10) are uniformly ultimately bounded. In other words the closed-loop error \( \mu(t) \) enters the compact set \( \Omega_d \) in finite time and will remain inside the set for all future times.

**V. Illustrative Example**

This section presents three simulation examples using a model for wing-rock dynamics.\(^{13}\) First, \( K \)-combined with \( e \)-modification (\( e\)-\( K \)-modification) are shown for different adaptation parameters. Second, time delay effects on \( \mathcal{H}_2^2 \)-combined with \( \sigma \)-modification (\( \sigma\)-\( K \)-modification) are given. Finally, \( \mathcal{H}_2^2K \)-AC results are presented by comparing them to \( \sigma\)-\( K \)-modification with and without the same time delay in the control channel.

Consider the following wing-rock dynamics

\[
\begin{align*}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= u(t) + \Delta(x(t))
\end{align*}
\]  

(44)

where \( \Delta(x(t)) = b_0 + b_1 x_1(t) + b_2 x_2(t) + b_3 x_1(t) x_2(t) + b_4 |x_2(t)| x_2(t) + b_5 x_2^3(t) \), \( x_1(t) \) and \( x_2(t) \) represent roll angle \( (\phi(t)) \) and roll rate \( (\dot{\phi}(t)) \), and \( b_{0,1,2,3,4,5} = [0.02314, 0.6918, -0.6245, 0.0095, 0.0214] \) which are derived from

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the aerodynamic coefficients. The control objective here is to eliminate the oscillations caused by the wing rock phenomenon by stabilizing the roll dynamics about the equilibrium condition \((x_1(t), x_2(t)) = (0, 0)\). The initial conditions are selected to be \(x_1(0) = \phi(0) = 6\) and \(x_2(0) = \dot{\phi}(0) = 3/\text{sec}\). The reference model is selected to be second order with a natural frequency of 0.5 rad/sec and a damping ratio of 0.707. Sigmoidal basis functions \(\beta_i(x(t)) = \frac{1}{1+e^{-x_i}}, i = 1, 2\) are used. The response of the baseline system and the reference model are shown in Figure 4.

![Figure 4. Reference model, aircraft, and baseline control responses](image)

Figures 5 through 16 show results with \(e\)-\(K\)-modification with the adaptation parameter settings summarized in Table 1. In Case 1, the adaptation gain is applied with zero stiffness gain. Figures 5 to 7 show that as the adaptation gain increases, both the amplitude and the frequency of the roll angle oscillation increase, as well as the control effort. Case 2 introduces \(K\)-modification with \(k = 100\) and \(T = 0.01\) second. Figures 8 to 10 show that the roll angle oscillation and the control effort dramatically decrease. In Case 3, the stiffness gain is increased with fixed adaptation gain. Figures 11 to 13 show that as the stiffness gain increases, the roll angle oscillation and its frequency decrease as well as the control effort decrease. Case 4 shows the effect that the value of \(T\) used in the update law in (9) has on performance with fixed adaptation and stiffness gain. Figures 14 to 16 show that the roll angle oscillation and its frequency as well as the control effort are significantly decreased for values of \(T\) in the range 0.01 - 0.1 seconds.

<table>
<thead>
<tr>
<th>Adaptation gain</th>
<th>(e)-modification</th>
<th>Stiffness gain</th>
<th>(T) [second]</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>(\lambda) fixed</td>
<td>(k)</td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>10, 25, 100</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>Case 2</td>
<td>10, 25, 100</td>
<td>100</td>
<td>250</td>
</tr>
<tr>
<td>Case 3</td>
<td>25</td>
<td>100</td>
<td>100, 250, 500</td>
</tr>
<tr>
<td>Case 4</td>
<td>25</td>
<td>100</td>
<td>250</td>
</tr>
</tbody>
</table>

Figures 17 and 18 show responses of \(\sigma\)-\(K\)-modification with time delay in the control input channel. Oscillations appear in Figure 16, where \(\sigma\)-modification alone is used with time a delay of 0.1 sec. Figure 18 shows that these oscillations are significantly reduced by employing \(K\)-modification.

Figures 19 and 20 show comparisons of \(\mathcal{H}_2^2\)\(K\)-\(AC\) and \(\sigma\)-\(K\)-modification with all adaptation parameters set to 1.0. in both cases. Figure 19 shows the response without time delay, while Figure 20 shows the result with a time delay of 0.1 sec. These results indicate that it is not as important to tune the adaptation parameters with \(\mathcal{H}_2^2\)\(K\)-\(AC\) as it is with \(\sigma\)-\(K\)-modification.
VI. Conclusion

$K$-modification based $\mathcal{H}_2$ adaptive control ($\mathcal{H}_2^2K$-AC) offers significant advantages in adaptive control design from the perspective of performance and robustness to time delay. It also requires less effort with regard to having to tune the parameters in the adaptive law, while at the same time it provides the ability to control both the damping and stiffness of the error response. $\mathcal{H}_2^2K$ adaptive control also significantly reduces oscillations due to time delay in actuation, as illustrated in the wing rock example.

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References

Figure 5. Adaptive control using $\gamma = 10, \lambda = 100, k = 0$

Figure 6. Adaptive control using $\gamma = 25, \lambda = 100, k = 0$

Figure 7. Adaptive control using $\gamma = 100, \lambda = 100, k = 0$
Figure 8. Adaptive control using $\gamma = 10$, $\lambda = 100$, $k = 250$, $T = 0.01$

Figure 9. Adaptive control using $\gamma = 25$, $\lambda = 100$, $k = 250$, $T = 0.01$

Figure 10. Adaptive control using $\gamma = 100$, $\lambda = 100$, $k = 250$, $T = 0.01$
Figure 11. Adaptive control using $\gamma = 25, \lambda = 100, k = 100, T = 0.01$

Figure 12. Adaptive control using $\gamma = 25, \lambda = 100, k = 250, T = 0.01$

Figure 13. Adaptive control using $\gamma = 25, \lambda = 100, k = 500, T = 0.01$
Figure 14. Adaptive control using $\gamma = 25, \lambda = 100, k = 250, T = 0.01$

Figure 15. Adaptive control using $\gamma = 25, \lambda = 100, k = 250, T = 0.05$

Figure 16. Adaptive control using $\gamma = 25, \lambda = 100, k = 250, T = 0.1$
Figure 17. Adaptive control with $\sigma$-modification and a time delay of 0.5 sec.

Figure 18. Adaptive control with $\sigma$-$K$-modification and a time delay of 0.5 sec.
Figure 19. $H_2^\sigma K$-AC and $\sigma$-$K$-modification with $\lambda = 1$, $\sigma = 1$, $k = 1$, without time delay

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Figure 20. $H_2^K$-AC and $\sigma$-$K$-modification with $\lambda = 1$, $\sigma = 1$, $k = 1$ with time delay 0.1 sec

(a) $H_2^K$-AC with $\sigma = 1$, $k = 1$, and time delay 0.1 sec

(b) $H_2$ Gain

(c) $\sigma$-$K$-mod. with $\lambda = 1$, $\sigma = 1$, $k = 1$, and time delay 0.1 sec