A derivative-free, delayed weight update law is developed for model reference adaptive control of continuous-time uncertain dynamical systems, without assuming the existence of constant ideal weights. Using a Lyapunov-Krasovskii functional it is proven that the error dynamics are uniformly ultimately bounded, without the need for modification terms in the adaptive law. Estimates for the ultimate bound and the exponential rate of convergence to the ultimate bound are provided. We also discuss employing various modification terms for further improving performance and robustness of the adaptively controlled system.

I. Introduction

In the past decades, numerous model reference adaptive control (MRAC) approaches have been proposed that deal with multivariable uncertain dynamical systems in continuous-time (see Refs. 1–10, and references therein). These approaches are derived based on Lyapunov theory and either assume or derive a weight update law in the form of an ordinary differential equation for the weight estimates. All these methods have in common the underlying assumption that there exists a constant, but unknown, ideal set of weights. Although this assumption seems reasonable and these MRAC architectures work well on many systems, in some failure modes they may require the use of unrealistically high adaptation gain, or may fail to achieve the desired level of performance in terms of failure recovery. MRAC laws that require high gain can excite unmodeled dynamics, typically exhibit an excessive amount of control activity\textsuperscript{11, 12}, amplify the effect of sensor noise, and are not sufficiently robust to time delay\textsuperscript{13}.

In this paper, we develop a derivative-free model reference adaptive control (DF-MRAC) law, which uses the information of delayed weight estimates and the information of current system states and errors. The proposed method is an extension of the iterative learning approach adopted in Ref. 15 for purposes of adaptive observer design. We relax the assumption of constant unknown ideal weights to the existence of time-varying weights, such that fast and possibly discontinuous variation in weights are allowed. The proposed derivative-free adaptive control law is advantageous for applications to systems that can undergo sudden change in dynamics, such as might be due to reconfiguration, deployment of a payload, or structural damage. We prove that the error dynamics are uniformly ultimately bounded using a Lyapunov-Krasovskii functional, without employing modification terms in the adaptive law. We consider constant unknown ideal weights as a special case and show that the state tracking error dynamics are asymptotically stable. Finally, we discuss employing various modification terms for further improving the performance and robustness of the adaptively controlled system.

The notation used in this paper is fairly standard. $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^T$ denotes transpose, and $(\cdot)^{-1}$ denotes inverse. Furthermore, we write $\lambda_{\min}(M)$ (resp., $\lambda_{\max}(M)$) for the minimum (resp., maximum) eigenvalue of the Hermitian matrix $M$, $\| \cdot \|$ for the Euclidean vector norm or for the Frobenius matrix norm, vec$(\cdot)$ for the column stacking operator.
diagonal matrix \([A, B]\) for a block diagonal matrix formed with matrices \(A\) and \(B\) on the diagonal, and \([a, b]\) denotes the open interval in \(\mathbb{R}\) from \(a\) to \(b > a\).

The organization of the paper is as follows. Section 2 provides preliminaries related to standard MRAC. Section 3 presents the DF-MRAC law and its stability properties. Section 4 discusses modifications to the DF-MRAC law intended to improve robustness. Section 5 treats a first-order example to illustrate the proposed approach. Finally, Section 6 summarizes the conclusions. A detailed application to a generic transport model aircraft can be found in Ref. 14.

II. Preliminaries

In this section we state standard results for the MRAC problem. Consider the controlled nonlinear uncertain dynamical system given by

\[
\dot{x}(t) = Ax(t) + Bu(t) + \Delta(x(t)) \tag{1}
\]

where \(x(t) \in \mathbb{R}^n\) is the state vector, \(u(t) \in \mathbb{R}^m\) is the control input, \(A \in \mathbb{R}^{n \times n}\) and \(B \in \mathbb{R}^{n \times m}\) are known matrices, and \(\Delta : \mathbb{R}^n \to \mathbb{R}^m\) is a matched uncertainty. Furthermore, we assume that the pair \((A, B)\) is controllable, the full state is available for feedback, and the control input \(u(t)\) is restricted to the class of admissible controls consisting of measurable functions.

The reference model is given by

\[
\dot{x}_m(t) = A_m x_m(t) + B_m r(t) \tag{2}
\]

where \(x_m(t) \in \mathbb{R}^n\) is the reference state vector, \(r(t) \in \mathbb{R}^r\) is a bounded piecewise continuous reference input, \(A_m \in \mathbb{R}^{n \times n}\) is Hurwitz, and \(B_m \in \mathbb{R}^{n \times r}\) with \(r \leq m\). Since \(r(t)\) is bounded, it follows that \(x_m\) is uniformly bounded for all \(x_m(0)\).

Assumption 2.1. The matched uncertainty in (1) can be linearly parameterized as

\[
\Delta(x(t)) = W^T \beta(x(t)) + \varepsilon(x(t)), \quad ||\varepsilon(x(t))|| \leq \varepsilon^* \tag{3}
\]

where \(W \in \mathbb{R}^{s \times m}\) is the unknown constant weight matrix, \(\beta : \mathbb{R}^n \to \mathbb{R}^s\) is a known vector of basis functions of the form \(\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^T \in \mathbb{R}^s\), and \(\varepsilon : \mathbb{R}^n \to \mathbb{R}^m\) is the residual error.

Consider the following feedback control law

\[
u(t) = u_n(t) - u_{\text{ad}}(t) \tag{4}\]

where \(u_n(t)\) is a nominal feedback control component given by

\[
u_n(t) = K_1 x(t) + K_2 r(t) \tag{5}\]

where \(K_1 \in \mathbb{R}^{m \times n}\) and \(K_2 \in \mathbb{R}^{m \times r}\) are nominal control gains such that \(A + BK_1\) is Hurwitz, and \(u_{\text{ad}}(t)\) is the adaptive feedback control component given by

\[
u_{\text{ad}}(t) = \dot{W}^T(t) \beta(x(t)) \tag{6}\]

where \(\dot{W}(t) \in \mathbb{R}^{s \times m}\) is an estimate of \(W\) satisfying the weight update law

\[
\dot{W}(t) = \gamma [\beta(x(t))] e^T(t) P B + \dot{W}_m(t), \quad \gamma > 0 \tag{7}\]

where

\[
e(t) \triangleq x(t) - x_m(t) \tag{8}\]

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is the state tracking error, \( P \in \mathbb{R}^{n \times n} \) is the positive-definite solution of the Lyapunov equation

\[
0 = A_m^T P + PA_m + Q \tag{9}
\]

for any \( Q = Q^T > 0 \), and \( \dot{W}_m(t) \in \mathbb{R}^{s \times m} \) is a modification term, e.g.

\[
\dot{W}_m(t) = -\sigma \dot{W}(t) \tag{10}
\]

for \( \sigma \)-modification term\(^1\), or

\[
\dot{W}_m(t) = -\sigma \|e(t)\|\dot{W}(t) \tag{11}
\]

for \( e \)-modification\(^2\), where \( \sigma \) is a positive fixed gain.

**Assumption 2.2.** \( A_m \) and \( B_m \) in (2) are chosen so that:

\[
A_m = A + BK_1 \tag{12}
\]

\[
B_m = BK_2 \tag{13}
\]

The dynamics in (1) can be written as

\[
\dot{x}(t) = A_m x(t) + B_m r(t) + B\dot{W}^T(t)\beta(x(t)) + B\varepsilon(x(t)) \tag{14}
\]

where

\[
\dot{W}(t) \triangleq W - \dot{W}(t) \tag{15}
\]

is the weight update error. The state tracking error and weight update error dynamics can likewise be written as:

\[
\dot{e}(t) = A_m e(t) + B\dot{W}^T(t)\beta(x(t)) + B\varepsilon(x(t)) \tag{16}
\]

\[
\dot{\dot{W}}(t) = \gamma[\beta(x(t))e^T(t)PB + \dot{W}_m(t)] \tag{17}
\]

Theorems that highlight the uniformly ultimate boundedness (UUB)\(^{17}\) of the closed-loop system errors given by (16) and (17) for the \( \sigma \)- and \( e \)-modification cases can be found in Refs. 1–3, 18. The typical Lyapunov candidate function used for the stability analysis of the adaptive law in (7) has the quadratic form

\[
V(e, \dot{W}) = e(t)^T Pe(t) + \frac{1}{\gamma} [e^T(t)\dot{W}(t)], \quad \gamma > 0 \tag{18}
\]

**III. Derivative-Free Adaptive Control**

The following assumption strengthens Assumption 2.1 by setting \( \varepsilon(x(t)) = 0 \), which can be justified under the assumption that time-variation is allowed in the unknown ideal weight matrix.

**Assumption 3.1.** The matched uncertainty in (1) can be linearly parameterized as

\[
\Delta(x(t)) = W^T(t)\beta(x(t)) \tag{19}
\]

where \( W(t) \in \mathbb{R}^{s \times m} \) is an unknown time-varying weight matrix that satisfies \( \|W(t)\| \leq w^* \) and \( \beta: \mathbb{R}^n \to \mathbb{R}^s \) is a vector of known functions of the form \( \beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^T \in \mathbb{R}^s \).

**Remark 3.1.** Assumption 3.1 expands the class of uncertainties that can be represented by a given set
of basis functions. That is, an adaptive law designed subject to Assumption 3.1 can be more effective than an adaptive law designed subject to Assumption 2.1 in canceling a wider class of uncertainties, due to the fact that time-variation is allowed in the unknown ideal weight matrix.

**Remark 3.2.** Assumption 3.1 does not place any restriction on the time derivative of the weight matrix.

The following theorem presents the main result of this paper.

**Theorem 3.1.** Consider the controlled nonlinear uncertain dynamical system given by (1) subject to Assumption 3.1. Consider, in addition, the feedback control law given by (4), with the nominal feedback control component given by (5) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has a derivative-free weight update law in the form

$$\dot{W}(t) = \Omega_1 \dot{W}(t - \tau) + \hat{\Omega}_2(t)$$  \hspace{1cm} (20)

where \( \tau > 0 \), and \( \Omega_1 \in \mathbb{R}^{s \times s} \) and \( \hat{\Omega}_2 : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{s \times m} \) satisfy:

$$0 \leq \Omega_1^T \Omega_1 < \kappa_1 I, \quad 0 < \kappa_1 < 1$$  \hspace{1cm} (21)

$$\hat{\Omega}_2(t) = \kappa_2 \beta(x(t)) e^T(t) P B, \quad \kappa_2 > 0$$  \hspace{1cm} (22)

with \( P \in \mathbb{R}^{n \times n} \) satisfying (9) for any symmetric \( Q > 0 \). Then, \( e(t) \) and \( \dot{W}(t) \) are UUB.

**Proof.** Using (20) and defining

$$\Omega_2(t) \triangleq W(t) - \Omega_1 W(t - \tau)$$  \hspace{1cm} (23)

where \( ||\Omega_2(t)|| \leq \delta^* \), \( \delta^* = w^*(1 + ||\Omega_1(t)||) \), the weight update error in (15) can be rewritten as:

$$\dot{W}(t) = \Omega_1 \dot{W}(t - \tau) + \Omega_2(t) - \hat{\Omega}_2(t)$$  \hspace{1cm} (24)

Using (24), the state tracking error dynamics in (16) under Assumption 3.1 becomes:

$$\dot{e}(t) = A_m e(t) + B \Omega_1 \dot{W}(t - \tau) - \hat{\Omega}_2(t) + \Omega_2(t)^T \beta(x(t))$$  \hspace{1cm} (25)

To show that the closed-loop system given by (24) and (25) is UUB, consider the Lyapunov-Krasovskii functional

$$V(e(t), \dot{W}_t) = e^T(t) Pe(t) + \rho \text{tr} \left[ \int_{t-\tau}^{t} \dot{W}^T(s) \dot{W}(s) ds \right]$$  \hspace{1cm} (26)

where \( \rho > 0 \) and \( \dot{W}_t \) represents \( \dot{W}(t) \) over the time interval \( t - \tau \) to \( t \). The directional derivative of (26) along the closed-loop system trajectories of (24) and (25) is given by

$$\dot{V}(e(t), \dot{W}_t) = -e^T(t) Q e(t) + 2e^T(t) P B [\Omega_1 \dot{W}(t - \tau)]^T \beta(x(t))$$

$$-2e^T(t) P B \hat{\Omega}_2(t)^T \beta(x(t)) + 2e^T(t) P B \Omega_2^T(t) \beta(x(t))$$

$$+ \rho \text{tr} \left[ -\xi \dot{W}^T(t) \dot{W}(t) + \eta \dot{W}^T(t) \dot{W}(t) - \dot{W}^T(t - \tau) \dot{W}(t - \tau) \right]$$  \hspace{1cm} (27)

where \( \eta = 1 + \xi \). It will be shown later that there is an optimal choice that one can make for \( \eta \) (see Remark 3.5). In what follows we impose the restriction \( \xi \geq 0 \).

Using (24) to expand the term \( \text{tr}[\eta \dot{W}^T(t) \dot{W}(t)] \) in (27) produces

$$\dot{V}(e(t), \dot{W}_t) = -e^T(t) Q e(t) + 2e^T(t) P B [\Omega_1 \dot{W}(t - \tau)]^T \beta(x(t)) - 2e^T(t) P B \Omega_2^T(t) \beta(x(t))$$

$$+ 2e^T(t) P B \hat{\Omega}_2(t)^T \beta(x(t)) + \rho \text{tr} \left[ -\xi \dot{W}^T(t) \dot{W}(t) - \dot{W}^T(t - \tau) \dot{W}(t - \tau) \right]$$

$$+ \eta \dot{W}^T(t - \tau) \Omega_1^T \Omega_1 \dot{W}(t - \tau) + \eta \hat{\Omega}_2^T(t) \hat{\Omega}_2(t) + \eta \Omega_2^T(t) \Omega_2(t)$$

$$- 2\eta \hat{\Omega}_2^T(t) \Omega_1 \dot{W}(t - \tau) + 2\eta \dot{W}^T(t - \tau) \Omega_1^T \Omega_2(t) - 2\eta \hat{\Omega}_2^T(t) \Omega_2(t)$$  \hspace{1cm} (28)
Next, consider the fact $a^Tb \leq \gamma a^Ta + \frac{1}{\gamma}b^Tb$, $\gamma > 0$, that follows from Young’s inequality\textsuperscript{15,16} for any vectors $a$ and $b$. This can be generalized to matrices as $\text{tr}(A^TB) = \text{vec}(A)^T\text{vec}(B) \leq \gamma \text{vec}(A)^T\text{vec}(A) + \frac{1}{\gamma\gamma}\text{vec}(B)^T\text{vec}(B)$, $\gamma > 0$, for any matrices $A$ and $B$ with appropriate dimensions. Using this, we can write

$$\text{tr}[2\eta\dot{W}^T(t-\tau)\Omega_1^T\Omega_2(t)] \leq \text{tr}[\gamma\dot{W}^T(t-\tau)\Omega_1^T\Omega_1\dot{W}(t-\tau)] + \text{tr}[\frac{\eta^2}{\gamma}\Omega_2^T(t)\Omega_2(t)], \gamma > 0$$ \quad (29)

Using (22) with $\kappa_2 \doteq 1/\rho\eta > 0$ for $\Omega_2(t)$, and substituting (29) in (28), it follows that

$$\dot{\mathcal{V}}(e(t), \dot{W}_i) \leq -e^T(t)Qe(t) - \kappa_2 e^T(t)PBB^TPe(t)\beta(x(t))x(t) - \rho \gamma \text{tr}[\dot{W}^T(t)\dot{W}(t)] - \rho \gamma \text{tr}[\dot{W}^T(t-\tau)[I - (\eta + \gamma)\Omega_1^T\Omega_1]\dot{W}(t-\tau)] + \rho(\eta + \frac{\eta^2}{\gamma})\text{tr}[\Omega_2^T(t)\Omega_2(t)]$$ \quad (30)

Using (21) with $\kappa_1 = \frac{1}{\eta + \gamma} < 1$ for $\Omega_1$ yields

$$\dot{\mathcal{V}}(e(t), \dot{W}_i) \leq -c_1\|e(t)\|^2 - c_2\||\dot{W}(t)\|^2 - c_3\||\dot{W}(t-\tau)\|^2 + d$$ \quad (31)

where the constants $c_1$, $c_2$, $c_3$, and $d$ are given by:

$$c_1 = \lambda_{\min}(Q) > 0$$

$$c_2 = \rho\gamma \geq 0$$

$$c_3 = \rho\lambda_{\min}(I - \kappa_1^{-1}\Omega_1^T\Omega_1) > 0$$

$$d = \rho(\eta + \frac{\eta^2}{\gamma})\delta^2 \geq 0$$ \quad (35)

where $\rho = \frac{1}{\kappa_2\eta}$. If $\xi > 0$, then since $\eta = \kappa_1^{-1} - \gamma = 1 + \xi$, $0 < \kappa_1 < 1$, and $\gamma > 0$, it follows that $\eta$ must line in the open interval $(1, 1/\kappa_1)$. Either $\|e(t)\| > \Psi_1$ or $\|\dot{W}(t)\| > \Psi_2$ or $\|\dot{W}(t-\tau)\| > \Psi_3$ renders $\dot{\mathcal{V}}(e(t), \dot{W}_i) < 0$, where $\Psi_1 = \sqrt{\frac{c_1}{c_2}}$, $\Psi_2 = \sqrt{\frac{c_1}{c_2}}$, and $\Psi_3 = \sqrt{\frac{c_1}{c_2}}$, or equivalently:

$$\Psi_1 = \delta^*/\lambda_{\min}(Q)$$

$$\Psi_2 = \Psi_1\sqrt{c_1/c_2}$$

$$\Psi_3 = \Psi_1\sqrt{c_1/c_2}$$

Hence, it follows that $e(t)$ and $\dot{W}(t)$ are UUB.

\[\square\]

The proposed adaptive control architecture is shown in Figure 1.

**Remark 3.3.** Letting $\ddot{W}_m(t) = 0$ and using a 1\textsuperscript{st} order Euler method for integration in (7), with $\tau_s$ being the step size results in:

$$\dot{W}(t) = \gamma \tau_s (\beta(x(t))e^T(t)PB) + \dot{W}(t - \tau_s)$$ \quad (39)

This form of weight update law is identical to the DF-MRAC law in (20), if $\Omega_1 = I$, $\kappa_2 = \gamma \tau_s$, and $\tau = \tau_s$, with the exception that the choice $\Omega_1 = I$ is not permitted in DF-MRAC. In DF-MRAC, $\tau$ is not necessarily equal to $\tau_s$ and $\Omega_1$ can be chosen, for example, as $\varsigma I$ where $0 < |\varsigma| < 1$.

**Remark 3.4.** The derivative-free weight update law given by (20) subject to (21) and (22) does not
require a modification term to prove the error dynamics, including the weight errors, are UUB.

**Corollary 3.1.** Under the conditions of Theorem 3.1, an estimate for the ultimate bound, for the case $\xi > 0$, is given by

$$r = \sqrt{\frac{\lambda_{\text{max}}(\tilde{P})}{\lambda_{\text{min}}(P)}(\Psi_1^2 + \tau \Psi_2^2)}$$  \hspace{1cm} (40)

where $\tilde{P} = \text{diag}[P, \rho]$.

**Proof.** Define $q(t) \triangleq [e^T(t), \tilde{v}(t, \tau)]^T$, where $\tilde{v}(t, \tau) \triangleq \text{tr}[\int_0^t \tilde{W}^T(s)\tilde{W}(s)ds]$, and let $B_r = \{q(t) : ||q(t)|| < r\}$, such that $B_r \subset D$ for a sufficiently large domain $D$. Furthermore, denote $\Omega_\alpha = \{q(t) \in B_r : q^T(t)\tilde{P}q(t) \leq \alpha\}$, $\alpha = \min_{||q(t)||=r} q^T(t)\tilde{P}q(t) = r^2\lambda_{\text{min}}(\tilde{P})$. Finally, let $\Omega_\beta = \{q(t) \in B_r : q^T(t)\tilde{P}q(t) \leq \beta\}$, where $\beta = \lambda_{\text{max}}(\tilde{P})(\Psi_1^2 + \tau \Psi_2^2)$. These sets are illustrated in Figure 2. Since

$$V(e(t), \tilde{W}_i) = q^T(t)\tilde{P}q(t)$$

$$= e^T(t)Pe(t) + \rho \text{tr}[\int_{-\tau}^t \tilde{W}^T(s)\tilde{W}(s)ds]$$

$$\leq \lambda_{\text{max}}(P)||e(t)||^2 + \rho \text{tr}[\int_{-\tau}^t ||\tilde{W}(s)||^2ds]$$

$$\leq \lambda_{\text{max}}(P)\Psi_1^2 + \rho \text{tr}[\int_{-\tau}^t \Psi_2^2ds]$$

$$= \lambda_{\text{max}}(P)\Psi_1^2 + \rho \tau \Psi_2^2$$

$$\leq \lambda_{\text{max}}(\tilde{P})(\Psi_1^2 + \tau \Psi_2^2)$$

(41)

it follows that $\Omega_\alpha$ is a positively invariant set if $\Omega_\beta \subset \Omega_\alpha$, and the minimum size of $B_r$ that ensures this condition has radius given by (40).
Corollary 3.2. Under the conditions of Theorem 3.1, the error trajectory approaches the ultimate bound exponentially in time according to
\[
||q(t)|| \leq \hat{k}||q(0)||e^{-c_r t}, \quad t < T
\] (42)
with a convergence rate given by:
\[
c_r = \frac{\tau}{2\lambda_{\max}(P)\Psi_1^2}, \quad \tau = \frac{\rho(\eta - 1)}{\lambda_{\min}(Q)} \tag{43}
\]

Proof. Choosing \( \tau \) so that \( c_2 = c_1 \tau \), and substituting the expressions for \( c_1 \) and \( c_2 \) in (32) and (33) results in the expression for \( \tau \) in (43). Then, from (31), we can write:
\[
\dot{V}(e(t), \tilde{W}_t) \leq -c_1||q(t)||^2 + d \tag{44}
\]
Define \( \hat{c} \triangleq d/\mu^2 \), where \( \mu \triangleq \sqrt{\Psi_1^2 + \tau \Psi_2^2} \). Then, when \( ||q(t)|| > \mu \), we have that
\[
\dot{V}(e(t), \tilde{W}_t) \leq -(c_1 - \hat{c})||q(t)||^2 \\
\leq -\hat{k}||q(t)||^2 \tag{45}
\]
where \( \hat{k} \triangleq c_1 - \hat{c} \). Now, \( c_1 = d/\Psi_1^2 \) and \( c_2 = d/\Psi_2^2 \), and since \( c_2/c_1 = \tau \), it follows that \( \Psi_2^2 = \tau \Psi_1^2 \), and therefore \( \hat{c} = d/(1 + \tau)\Psi_1^2 \). Finally, since \( k_1||q(t)||^2 \leq V(e(t), \tilde{W}_t) \leq k_2||q(t)||^2 \), where \( k_1 = \lambda_{\min}(\tilde{P}) > 0 \), \( k_2 = \lambda_{\max}(\tilde{P}) > 0 \), and \( \dot{V}(e(t), \tilde{W}_t) \leq -\hat{k}||q(t)||^2 \), it follows from Corollary 5.3 of Ref. 20 that
\[
||q(t)|| \leq \hat{k}||q(0)||e^{-c_r t}, \quad t < T
\] (46)
where \( \hat{k} = \sqrt{\lambda_{\max}(\tilde{P})/\lambda_{\min}(\tilde{P})} \), and \( c_r = (\hat{k}/2k^2) = \tau/2(1 + \tau)\lambda_{\max}(\tilde{P})\Psi_1^2 \).

Remark 3.5. For the most common case in which \( \lambda_{\max}(\tilde{P}) > \rho \), there exists:
\[
\eta^* = (-\varphi_1 + \sqrt{\varphi_1^2 - 4\varphi_2})/2 \tag{47}
\]
\[
\varphi_1 \triangleq -\frac{2}{1 + \kappa_2 \lambda_{\min}(Q)} \tag{48}
\]
\[
\varphi_2 \triangleq \frac{\kappa_1 - \kappa_2 \lambda_{\min}(Q)}{\kappa_1(1 + \kappa_2 \lambda_{\min}(Q))} \tag{49}
\]
where $\eta^*$ is the value of $\eta$ such that the convergence rate $c_r$ in (43) attains a maximum on the open interval $\{1, 1/\kappa_1\}$. Figures 3 and 4 show plots of the normalized ultimate bound ($r/\delta^*$) in (40) versus convergence rate $c_r$ in (43) for $\lambda_{\text{max}}(\tilde{P}) = 2$, and $\lambda_{\text{min}}(\tilde{P}) = \lambda_{\text{min}}(Q) = 1$, and $\eta = \eta^*$. Figure 3 shows that increasing $\kappa_2$ increases the ultimate bound, but it also increases the convergence rate. The fact that the estimate for the ultimate bound increases with the adaptation gain has been previously noted in Refs. 7 and 12. Figure 3 also shows that increasing $\kappa_1$ from 0.01 to 0.03 reduces the ultimate bound for any given value of $\kappa_2 \in [1, 10]$. However, if we increase $\kappa_1$ from 0.1 to 0.3, then Figure 4 shows that the trend in Figure 3 reverses for values $\kappa_2 \in [1, 10]$. This calculation can be repeated for any given set of parameters in (40) and (42) as a guide to choosing the design parameters of DF-MRAC.

For the case of constant ideal weights in Assumption 3.1, Theorem 3.1 specializes to the following theorem. In this case, we assume that the uncertainty is structured. That is, the vector of known functions in (19) represent the vector of known basis functions.
Theorem 3.2. Consider the controlled nonlinear uncertain dynamical system given by (1) subject to Assumption 3.1, where $W \in \mathbb{R}^{s \times m}$ is an unknown constant weight matrix. Consider, in addition, the feedback control law given by (4), with the nominal feedback control component given by (5) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has the derivative-free weight update law in the form (20) and (22), where $\Omega_t = I$. Then, $e(t)$ and $\dot{W}(t)$ approach a subspace in these error variables in which $e(t) = 0$ and $\dot{W}(t)\beta(x(t)) = 0$.

Proof. The result follows directly from the proof of Theorem 3.1 by choosing $\Omega_1 = I$. In this case, $\delta^* = 0$ due to $\Omega_2(t) = 0$ in (23), which follows from the fact that the ideal weights are constant, and $\Omega_1 = I$. Then, the inequality in (29) is not needed since the left hand side vanishes. In this case, $\kappa_1 = \frac{1}{\eta} = 1, \xi = 0$, $c_2 = 0$ in (33), $c_3 = 0$ in (34), and $d = 0$ in (35). Therefore, it follows from (31) that the entire error space is invariant. Let $\mathcal{E}$ denote the set of points in this space where $\mathcal{V}(e(t), \dot{W}_i) = 0$. From (31) all points in Let $\mathcal{E}$ lie in a subspace where $e(t) = 0$ and it follows from LaSalle’s Theorem that all solutions in the error space approach the largest invariant set $\mathcal{M}$ in $\mathcal{E}$. Now $e(t) = 0$ implies that $\dot{\Omega}_2(t) = 0$ and $\dot{e}(t) = 0$. Assuming $B$ has full column rank, then (25), with all of the above taken together, implies that $\mathcal{M}$ is comprised of all points in the error space in which $e(t) = 0$ and $\dot{W}(t)\beta(x(t)) = 0$.

Remark 3.6. The system is said to be sufficiently excited if $r(t)$ is such that the conditions $e(t) = 0$, $\dot{W}(t)\beta(x(t)) = 0$ admit only the solution $\dot{W}(t) = 0$ in the limit $t \to \infty$. It is straightforward to show that this amounts to the standard MRAC condition for persistency of excitation.

IV. Modifications to Derivative-Free Adaptive Control

Although the derivative-free weight update law does not require a modification term to prove the error dynamics are UUB, one may wish to employ a modification term in order to improve performance or robustness of the system. The following theorem extends Theorem 3.1 to a general form of modified DF-MRAC.

Theorem 4.1. Consider the controlled nonlinear uncertain dynamical system given by (1) subject to Assumption 3.1. Consider, in addition, the feedback control law given by (4), with the nominal feedback control component given by (5) subject to Assumption 2.2, and with the adaptive feedback control component given by (6) that has a derivative-free weight update law in the form given by (20), where $\tau > 0$ is a time delay design value, and $\Omega_t \in \mathbb{R}^{s \times s}$ and $\Omega_2(t) = \Omega_2(x(t), e(t)), \Omega_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s \times m}$, satisfy:

$$0 < \Omega_1^T \Omega_1 < \kappa_1 I, \quad 0 < \kappa_1 < 1$$

$$\dot{\Omega}_2(t) = \kappa_2 [\beta(x(t))e^T(t)PB - \kappa_m S(t) \dot{W}(t)], \quad \kappa_m > 0$$

where $\kappa_m$ is the modification gain, $P \in \mathbb{R}^{n \times n}$ satisfies

$$0 = A^T_m P + PA_m - \kappa_m^2 PBB^T P + Q$$

for any symmetric $Q > 0$, and $S(t) \in \mathbb{R}^{s \times s}$ satisfies $\|S(t)\| < s^*$. Then, $e(t) = x(t) - x_m(t)$ and $\dot{W}(t) = W(t) - \dot{W}(t)$ are UUB.

Proof. The result follows directly from the proof of Theorem 3.1, with (32) - (35) changed to:

$$c_1 = \lambda_{\min}(Q) > 0$$

$$c_2 = \rho\xi + s^*\kappa^* > 0$$

$$c_3 = \rho\lambda_{\min}(I - [1 + \eta + \gamma]\Omega_1^T \Omega_1) > 0$$

$$d = \rho(1 + \eta + \frac{\eta^2}{\gamma})\delta^2 + \omega^* s^* \kappa^* > 0$$

where $\kappa^* = 1 + 2\kappa_m + (1 + \xi)^{-1}\kappa_m^2 > 0$. 

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Table 1. DF-MRAC laws for various modification terms

<table>
<thead>
<tr>
<th>Modification</th>
<th>DF-MRAC Law for (-1 &lt; \varphi_1 &lt; 1, \varphi_2 &gt; 0, \dot{\varphi}_2 &gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original (20)</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB]).</td>
</tr>
<tr>
<td>(\sigma^1)</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}_2 \dot{\hat{W}}(t)]).</td>
</tr>
<tr>
<td>(e^2)</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}_2 |e(t)| \dot{\hat{W}}(t)]).</td>
</tr>
<tr>
<td>ALR(^{22})</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}_2 \beta_2(x(t)) \beta_2^T(x(t)) \hat{W}(t)]).</td>
</tr>
<tr>
<td>Q(^{23,24})</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}<em>2 (\hat{W}^T q(t) - c(t)) q(t)]); (q(t) \triangleq \int</em>{t_d}^{t} \beta(x(s)) ds, t_d &gt; 0, c(t) \triangleq e_n(t) - e_n(t_d) - \int_{t_d}^{t} B^T A_m \times A_m e(s) ds + \int_{t_d}^{t} u_{ad}(s) ds), and (e_n(t)) is the (n)-th component of the vector (e(t)), and (t_d &gt; 0).</td>
</tr>
<tr>
<td>Optimal(^{25})</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}_2 \beta(x(t)) \beta(x(t))^T \hat{W}(t) B^T P A_m^{-1} B]).</td>
</tr>
<tr>
<td>CMRAC(^{26})</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}_2 \bar{\beta}_2(x(t)) \bar{e}_Y(t)]); where (\bar{\beta}(x(t)) = \mathcal{L}^{-1}{\frac{1}{\sqrt{\pi x^2(t)}}} \beta(x(t))), where (\mathcal{L}^{-1}) denotes the Laplace operator, and (e_Y(t)) is the predictor estimation error defined in Ref. 26.</td>
</tr>
<tr>
<td>(K^{27})</td>
<td>(\hat{W}(t) = \varphi_1 \hat{W}(t) + \varphi_2 [\beta(x(t)) e^T PB - \dot{\varphi}<em>2 \int</em>{t_d}^{t} \hat{W}(s) ds]); (\text{where } t_d &gt; 0).</td>
</tr>
</tbody>
</table>

Remark 4.1. If \(S(t)\) is not assumed to be bounded by \(s^*\), e.g. \(S(t) = ||e(t)||\) in the case of \(e\)-modification\(^2\), then one can show that \(e(t)\) and \(\hat{W}(t)\) are UUB by applying a projection operator\(^4\) to the weight estimates given by (20).

Remark 4.2. In the case of \(\sigma\)-modification\(^1\), we have \(S(t) = I\) in (51). Recently an adaptive loop recovery (ALR) modification term\(^{22}\) was proposed with the objective of recovering the loop transfer properties of a chosen reference system. ALR modification can be introduced by letting \(S(t) = \beta_2(x(t)) \beta_2^T(x(t))\), where \(\beta_2(x(t)) \triangleq \frac{d\beta(x(t))}{dt}\) \(\in \mathbb{R}^{n \times n}\) and in this case \(s^* = 1\). Table 1 summarizes DF-MRAC laws for this and other modification terms as well.

V. First Order System Example

In this section we compare the standard MRAC law given by (7) with the proposed DF-MRAC law given by (20) on a simple model for the rolling dynamics of an aircraft\(^{28}\). For this purpose, consider the scalar dynamics

\[
\dot{x}(t) = L_p x(t) + L_\delta [u(t) + w(t)x(t)]
\] (57)

where \(x(t)\) represents roll rate, \(u(t)\) represents aileron deflection, \(L_p = -1\), \(L_\delta = 1\), and \(w(t)\) is a square wave having an amplitude of 1.0, and a period of 5.0 seconds. While aircraft dynamics do not behave this manner, their stability derivatives can undergo a sudden change in the event of damage to the airframe. So this example should be regarded in this context. In both the MRAC and DF-MRAC architectures, we choose as a reference model \(A_m = -2\) and \(B_m = 2\). Note that for this example \(\varepsilon(x(t)) = 0\), so we may use
the MRAC law given by (7) without a modification term ($\sigma = 0$), and choose $\beta(x(t)) = x(t)$. Furthermore, we considered two different adaptation gains, $\gamma = 10^2$ and $\gamma = 10^4$. For the DF-MRAC law given by (20), we set $\tau = 0.01$ seconds, $\Omega_1 = 0.5$ which satisfies (21), and $\kappa_2 = 200$ in (22). $Q = 2$ was used in (9) to solve for $P$ in both architectures. Figures 5–14 present the results.

Figure 5 shows the performance of the nominal control design without adaptation. In Figure 6, the standard MRAC architecture is used with $\gamma = 10^2$ and $\gamma = 10^4$, respectively. Tracking performance is not improved by increasing the adaptation gain beyond $10^2$, and increasing gain causes high frequency oscillations in the control response that would be unacceptable in a real system. Figure 7 shows the case in which the DF-MRAC adaptive law in (20) was used with $\kappa_2 = 200$. It can be seen that tracking performance is excellent, and the control time history is reasonable, and in this case the estimated weight is close to the ideal value.

As a variation of the previous example, the ideal weight history was changed to a sinusoidal function in which the frequency was varied from 0.5 rad/s up to 50 rad/s. The results are shown in Figures 8–11. Comparing Figures 8 and 9, note that the adaptive controller does not significantly improve the response at the low setting for adaptation gain, and gives an even worse response when using the high setting for adaptation gain. Figures 10 and 11 show that the DF-MRAC case gives an excellent response, and the estimated weight is very close to the true weight, even at the high frequency end (see Figure 11). Inspired by this result we decided to try a case in which $w(t)$ is a band-limited white noise signal. The associated results are shown in Figures 12–14. In particular, Figure 14 shows that the estimated weight is remarkably close to $w(t)$.

![Figure 5. Responses with nominal controller for the square wave ideal weight.](image-url)
Figure 6. Responses with standard MRAC using $\gamma = 10^2$ and $\gamma = 10^4$.

Figure 7. Responses with DF-MRAC for the square wave ideal weight.
Figure 8. Responses with nominal controller for a sinusoidal ideal weight.

Figure 9. Responses with standard MRAC using $\gamma = 10^2$ and $\gamma = 10^4$. 
Figure 10. Responses with DF-MRAC for a sinusoidal ideal weight.

Figure 11. Expanded view of the estimate of the ideal weight.
Figure 12. Responses with nominal controller for a band limited white noise ideal weight.

Figure 13. Responses with DF-MRAC for a band limited white noise ideal weight.
VI. Conclusions

This paper presents a derivative-free model reference adaptive control law for nonlinear uncertain dynamical systems. The system error signals, including the state tracking error and the weight update error, are proven to be uniformly ultimately bounded using a Lyapunov-Krasovskii functional without the introduction of modification terms, for the case when the time-variation is allowed in the uncertain ideal weights. We have also shown that the errors approach the ultimate bound exponentially in time. The approach is particularly useful for situations in which the nature of the system uncertainty cannot be adequately represented by a set of basis functions with unknown by constant weightings, or for situations in which the ideal weights can undergo a discontinuous change. In addition to showing ultimate boundedness, we have provided an analysis that shows how the design parameters in the adaptive law can influence the ultimate bound, and the exponential rate of convergence to the bound.

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References


