A Parameter Dependent Riccati Equation Approach to Output Feedback Adaptive Control

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A parameter dependent Riccati equation approach is taken to analyze the stability properties of an output feedback adaptive control law design. The adaptive controller is intended to augment a nominal, fixed gain, observer based output feedback control law. Although the formulation is in the setting of model following adaptive control, the realization of the adaptive controller does not require implementing the reference model. In this regard, the cost of implementing the adaptive controller above that of a fixed gain control law is far less than that of other methods. The error signals are shown to be uniformly ultimately bounded and an expression for the ultimate bound is provided. The control design process and theoretical results are illustrated using a model for wing-rock dynamics.

I. Introduction

Research in adaptive output feedback control of uncertain nonlinear systems is motivated by the many emerging applications that employ novel actuation devices for active control of flexible structures and fluid flows. These applications include actuators such as piezo-electric films and synthetic jets, which are typically nonlinearly coupled to the plant dynamics they are intended to control. Models for these applications vary from accurate low frequency models to models that crudely approximate the true dynamics even at low frequencies. Examples of applications include active damping of flexible structures, control of aeroservoelastic aircraft, and active control of flows. Adaptive control can be used to satisfy performance requirements in the presence of large scale parameter uncertainty, and improved safety in the event of actuator failure.

There have been two approaches to dealing with adaptive output feedback control. One approach is based on a state estimation, whereas the other avoids the use of a state observer. Esfandiari and Khalil 1 introduced a high gain observer for the reconstruction of the unavailable states. Kristic, Kanellopoulos, Kokotovic 2 and Marino, Tomei 3 have presented backstepping-based approaches to adaptive output feedback control of uncertain systems, which are affine with respect to unknown parameters. They use derivatives of inputs and outputs directly, which is not desirable if there is noise in the system. Kim and Lewis 4 suggested a neural network (NN) in the observer structure. Adaptive output feedback control using a high gain observer and radial basis function NNs has also been proposed by Seshagiri and Khalil 5 for nonlinear systems represented by input-output models. Another method involving design of an adaptive observer using function approximators and backstepping control was given by Choi and Farrell 6 for a limited system that can be transformed to output feedback form in which nonlinearities depend on measurements only. In Calise, Hovakimyan, and Idan, 7 a direct adaptive output feedback control design procedure was developed for uncertain nonlinear systems without state estimation but with a stable low pass filter to satisfy a strictly positive real (SPR) condition. Their approach requires that the input vector to the NN be composed of current and past input/output data. However, their approach is limited to single input systems. Hovakimyan, Nardi, Calise, and Kim 8 considered adaptive output feedback control of uncertain nonlinear systems with an error observer instead of a state observer. Volyansky, Haddad, and Calise 9 introduced Q-modification to the adaptive output feedback control problem. Yucelen, Haddad, and Calise 10 used a nonminimal state space realization, which constructs a higher input-output equivalent system from the original system. Lavretsky 11 introduced an adaptive output feedback design using asymptotic properties of LQG/LTR controllers, that asymptotically satisfies the SPR condition. All of the above approaches, with the exception

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of that in Ref. 11, significantly increase the level of complexity in adaptive control design. Our approach is similar in spirit to that of Ref. 11, with the exception that it is less restrictive in that we do not attempt to satisfy an SPR condition, which naturally leads to a formation that relies on the existence of a positive definite solution of a parameter dependent Riccati equation. Therefore, our approach is applicable to non-minimum phase systems.

In some instances, applying an adaptive controller implies replacement of an existing control system. However, it is highly desirable to consider an adaptive approach that can be implemented in a form that augments an existing controller, rather than replace it with an entirely new control system. This rationale has been a main driving force for applying adaptive control in Ref. 12–16.

This paper presents an adaptive output feedback approach assuming that a state observer is employed in the nominal controller design. The observer design is modified and employed in the adaptive part of the design in place of a reference model. This is combined with a novel adaptive weight update law. The weight update law ensures that estimated states follow both the reference model states and the true states so that both state estimation errors and state tracking errors are bounded. Although the formulation is in the setting of model following adaptive control, the realization of the adaptive controller uses the observer of the nominal controller in place of the reference model to generate an error signal. Thus the only components that are added by the adaptive controller are the realizations of the basis functions and the weight adaptation law. The realization is even less complex than that of implementing a model reference adaptive controller in the case of state feedback.

The stability analysis employs a Lyapunov candidate function that entails the solution of a parameter dependent Riccati equation (rather than a Lyapunov equation) to show that all error signals are uniformly ultimately bounded (UUB). It is shown how the upper limit for the Riccati equation parameter is employed in the design of the adaptive law, and also influences the ultimate bounds for the state estimate error and the adapted weight error. The design procedure is illustrated with a second-order wing-rock dynamics model.

The following notations are used: $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^T$ denotes transpose, and $(\cdot)^{-1}$ denotes inverse, vec$(A)$ is the vector form of a matrix $A$ in which the columns of $A$ are stacked into a vector, diag$(A,B)$ is a block diagonal matrix formed with matrices $A$ and $B$ on the diagonal, and $I_n$ is an $n \times n$ identity matrix. tr$(A)$ is the trace of matrix $A$, $\cdot$ denotes vector norm, $\parallel \cdot \parallel$ denotes matrix maximum singular value, $\parallel \cdot \parallel_F$ denotes Frobenius norm, $\lambda_{\text{min}}(A)$ and $\lambda_{\text{max}}(A)$ denote the minimum and maximum eigenvalues of matrix $A$, respectively.

The paper is organized as follows. Section II provides preliminaries related to model reference adaptive control (MRAC), the control objective, a standard Luenberger observer, and a weight update law for the output feedback adaptive control design. Section III proves a theorem and corollaries that state the conditions under which the error signals are UUB and provide expressions for the ultimate bounds. Section IV provides a numerical example to illustrate the proposed approach. Conclusions are given in Section V.

## II. Output Feedback Adaptive Control Design

Consider the uncertain system given by:

$$\begin{align*}
\dot{x}(t) &= Ax(t) + B[u(t) + \Delta(x(t))] \\
y(t) &= Cx(t)
\end{align*}$$

(1)

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{m \times n}$ are known system matrices; $x(t) \in \mathbb{R}^n$ is the state vector; $u(t) \in \mathbb{R}^m$ is the control input; $\Delta(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the unknown matched uncertainty; $y(t) \in \mathbb{R}^p$, $p \geq m$ is the measured output vector, $m$ elements of which are to be regulated; and the triple $\{A,B,C\}$ is minimal.

**Remark 2.1.** The system given by (1) assumes that the control input vector and the regulated output vector have the same dimension. For the case when the dimension of the control input vector is larger than the dimension of the regulated output vector, due to redundant actuation, one can use matrix inverse and pseudo-inverse approaches, constrained control allocation, pseudo-controls, and daisy chaining\textsuperscript{17–19} to reduce the dimension of the control input vector to the dimension of the regulated output vector. Furthermore, the system can have a sensed output vector.
\[
y_y(t) = C_x x(t)
\]
where \(y_y(t) \in \mathbb{R}^p\), \(C_x \in \mathbb{R}^{p \times n}\), \(p \geq m\) such that the elements of \(y(t)\) are a subset of the elements of \(y_y(t)\).

Consider the following nominal control law
\[
u_n(t) = -K_x \hat{x}(t) + K_r r(t)
\]
where \(K_x \in \mathbb{R}^{m \times n}\) and \(K_r \in \mathbb{R}^{m \times m}\) with \(m \leq p\) are given feedback and feedforward gain matrices, respectively, \(\hat{x}(t)\) is an estimate of \(x(t)\) in an observer based design, and \(r(t) \in \mathbb{R}^m\) is a bounded reference command. Define the reference model:
\[
\begin{align*}
\dot{x}_m(t) &= A_m x_m(t) + B_m r(t) \\
y_m(t) &= C x_m(t)
\end{align*}
\]
where \(x_m(t) \in \mathbb{R}^n\) is the model state, \(y_m(t) \in \mathbb{R}^p\) is the model output, \(A_m = A - BK_x\) is Hurwitz by design, and \(B_m = BK_r\). The gains of the nominal control law are designed for the system in (1) assuming full state feedback, with \(\Delta(x(t)) = 0\), so that a subset of the elements in \(y_m(t)\) tracks \(r(t)\) to within some set of specifications on both the transient and steady state performance.

**Assumption 2.1.** The matched uncertainty in (1) can be linearly parameterized as
\[
\Delta(x) = W^T \beta(x) + \varepsilon(x), \quad \forall x \in D
\]
where \(W \in \mathbb{R}^{s \times m}\) is an unknown constant ideal weight matrix that satisfies \(||W|| = \omega^*\), \(\beta(\cdot) : \mathbb{R}^n \to \mathbb{R}^s\) is a known basis vector of the form \(\beta(x) = [\beta_1(x), \beta_2(x), \ldots, \beta_s(x)]^T\) with \(|\beta(x)| \leq \tilde{\beta}\), and \(\varepsilon(x)\) is the residual error satisfying \(|\varepsilon(x)| < \tilde{\varepsilon}\) for a sufficiently large domain \(D \subset \mathbb{R}^n\).

**Assumption 2.2.** The basis function \(\beta(\cdot)\) in (5) is Lipschitz continuous on \(D\)
\[
|\beta(x) - \beta(\tilde{x})| \leq L_{\beta}|x - \tilde{x}|, \quad \forall x, \tilde{x} \in D
\]

The adaptive control objective is to design a control law \(u(\cdot)\) for the dynamics in (1) so that the output \(y(t)\) tracks the reference model output \(y_m(t)\) with bounded error. The nominal control law \(u_n(t)\) given by (3) is augmented by an adaptive control \(u_{ad}(t)\)
\[
u(t) = u_n(t) - u_{ad}(t)
\]
where
\[
u_{ad}(t) = \hat{W}^T(t) \beta(\hat{x}(t))
\]
where \(\hat{x}(t)\) is an estimate of \(x(t)\) obtained using a state observer given by:
\[
\begin{align*}
\dot{\hat{x}}(t) &= A \hat{x}(t) + B u_n(t) + L[y(t) - \hat{y}(t)] \\
\dot{\hat{y}}(t) &= C \hat{x}(t)
\end{align*}
\]
where \(L \in \mathbb{R}^{n \times p}\) is an observer gain matrix designed such that \(A - LC\) is Hurwitz. The state observer in (9) is regarded as a part of the nominal control design. However, our viewpoint is that \(L\) may be altered for purposes of adaptively augmenting the nominal controller. Denote the state estimation error, the estimated state tracking error, the tracking error, and the weight estimate error as:
\[
\begin{align*}
\hat{e}(t) &= x(t) - \hat{x}(t) \\
\hat{e}(t) &= \hat{x}(t) - x_m(t) \\
e(t) &= x(t) - x_m(t) \\
\hat{W}(t) &= W - \hat{W}(t)
\end{align*}
\]
The dynamics for the state estimation error, $\tilde{x}(t)$, and the estimated state tracking error, $\hat{e}(t)$, are described as:

$$\dot{\tilde{x}}(t) = A_e \tilde{x}(t) + B \tilde{\Delta}(t)$$

$$\dot{\hat{e}}(t) = A_m \hat{e}(t) + LC \tilde{x}(t)$$

(11)

where $A_e = A - LC$ and $\tilde{\Delta}(t) = \Delta(x) - u_{ad}(t)$.

Consider the parameter dependent Riccati equation

$$0 = A_e^T P + PA_e + Q_0 + \mu NN^T$$

$$N = C^T - PB$$

(12)

where $Q_0 \geq 0$ and $0 < \mu < \bar{\mu}$ defines the set within which there exists a positive definite solution for $P$. Note that $N = 0$ corresponds to the situation in which $\{A_e, B, C\}$ is positive real. The asymptotic approach in Ref.11 is aimed at reducing the size of $N$, whereas in our approach $N \neq 0$ is treated as giving rise to a parameter dependent Riccati equation. Furthermore, consider the following weight update law

$$\dot{\hat{W}}(t) = \gamma \left[ \beta(\tilde{x}) \tilde{y}^T(t) - \left( \sigma I + \beta(\tilde{x}) \beta(\tilde{x})^T / 2\mu \right) \hat{W}(t) \right]$$

(13)

where $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}$ are positive adaptation gains, and $\tilde{y}(t) = y(t) - \hat{y}(t)$.

Remark 2.2. The structure of the adaptive law in (13) is novel in that it contains an additional term depending on $\mu$, the parameter in the Riccati equation in (12).

Figure 1 shows the overall adaptive output feedback control system architecture. It should be noted that the reference model is not used in the adaptive output feedback design. The observer serves as the reference model. Its dynamics are the same as the reference model if $u_{ad}(t)$ cancels $\Delta(x)$, and in this case the observer error transient $\tilde{y}(t)$ goes to zero. So in the end the components that are added to the baseline controller design in order to realize the adaptive control consist of computing the basis functions, forming the adaptive law, and integration of $\dot{\hat{W}}(t)$.
Remark 2.3. If $N = 0$ in (12) then it follows that we have:

$$
0 = A^T e P + P A e + Q_0 \\
0 = C^T - P B
$$

(14)

which implies that the transfer function associated with the system $G(s) = C(sI_n - A_e)^{-1}B$ is positive real. In this case (12) reduces to a Lyapunov equation associated with the state estimation error dynamics in (11), which is usually employed in the stability analysis of adaptive system, and $\bar{\mu} = \infty$. This suggests that for the purpose of adaptive control design, when $m > 1$, it is advantageous to define a new measurement by taking a linear combination of the existing measurements

$$
y_o(t) = My(t) = MCx(t) = C_o x(t)
$$

(15)

where $M \in \mathbb{R}^{m \times m}$ is a norm preserving transformation that minimizes a norm measure of $N_o$ where

$$
N_o \equiv C o^T - PB
$$

(16)

with $P_o$ defined as the value of $P$ that satisfies the Lyapunov equation in (14). Taking the Frobenius norm as a measure, it can be shown that the solution for $M$ that minimizes $\|N_o\|_F$ subject to the constraint $\|MC\|_F = \|C\|_F$ is given by:

$$
M = k M_o \\
M_o^T = (C C^T)^{-1} C P_o B \\
k = \frac{\|C\|_F}{\|M_o C\|_F}_F
$$

(17)

Remark 2.4. $P > 0$ for $\mu = 0$ and $P$ depends continuously on $\mu$. Therefore the existence of $\bar{\mu} > 0$ is assured.

If $A_m$ has no repeated eigenvalues and the observer gain matrix, $L$, is designed using pole placement such that $\lambda(A_e) = k \lambda(A_m)$, then for a given $\mu < \bar{\mu}$ we can denote $P(k)$ as the corresponding positive definite solution for $P$. In this case we can state the following Lemma.

Lemma 2.1. Lim$_{k \rightarrow \infty} P(k) = 0$.

Proof: Let $T_m$ and $T_e$ denote diagonalizing transformations for $A_m$ and $A_e$, so that:

$$
A_m = T_m \text{diag} \{ \lambda(A_m) \} T_m^{-1} \\
A_e = k T_e \text{diag} \{ \lambda(A_m) \} T_e^{-1}
$$

(18)

and let $T_s = T_e T_m^{-1}$. Then

$$
A_e = k T_s A_m T_s^{-1} \equiv k A_s
$$

(19)

Defining $P_k \equiv kP(k)$ and $N_k \equiv C^T - k^{-1} P_k B$, (12) can be written as

$$
0 = A_s^T P_k + P_k A_s + Q_0 + \mu N_k N_k^T
$$

(20)

Taking the lim$_{k \rightarrow \infty}$ of (20), it follows that

$$
0 = A_s^T P_\infty + P_\infty A_s + Q_0 + \mu C C^T
$$

(21)

Since the solution of (21) ($P_\infty > 0$) is finite, it follows that

$$
\lim_{k \rightarrow \infty} P(k) = \lim_{k \rightarrow \infty} \frac{P_k}{k} = 0
$$

(22)

The next lemma shows that (12) can be solved for $P > 0$ using Potter’s method.$^{20}$ This also means that $\bar{\mu}$ can be determined by searching for the boundary value that results in a failure of the algorithm to converge. We employ the notation Ric$(\cdot)$ and dom(Ric) as defined in Ref.21.
Lemma 2.2. Define the Hamiltonian matrix

\[ H \equiv \begin{bmatrix} A_e - \mu BC & \mu R \\ -Q & -(A_e - \mu BC)^T \end{bmatrix} \]  

where \( Q \equiv Q_0 + \mu CC^T \) and \( R \equiv BB^T \). Then for all \( 0 < \mu < \bar{\mu} \), \( H \in \text{dom}(\text{Ric}) \) and \( P = \text{Ric}(H) \).

Proof: The proof follows from Lemmas 3 and 4 as stated in Ref.21.

Remark 2.5. It can and often does happen that the Ricatti equation in (21) will have more than one positive definite solution. However, since all we require is that a solution exists and that a unique solution can be reliably computed using Potter’s method, the fact that other solutions exist seems to have no bearing on the design approach. Therefore the implication of multiple solutions is not pursued in this paper.

The uncertainty estimation error \( \tilde{\Delta}(t) \) can be written as

\[ \tilde{\Delta}(t) = \Delta(x) - u_{ad}(t) \]
\[ = W^T \beta(x(t)) + \varepsilon(x) - \dot{\hat{W}}^T(t) \beta(\hat{x}(t)) \]
\[ = \dot{W}^T(t) \beta(\hat{x}(t)) + W^T (\beta(x(t)) - \beta(\hat{x}(t))) + \varepsilon(x) \]

Hence the system error dynamics (11) can be rewritten as:

\[ \dot{x}(t) = A_x \bar{x}(t) + B [ \dot{W}(t) \beta(\hat{x}(t)) + W^T (\beta(x(t)) - \beta(\hat{x}(t))) + \varepsilon(x) ] \]
\[ \dot{\bar{e}}(t) = A_m \bar{e}(t) + LC \tilde{x}(t) \]

III. Boundedness of the Error Dynamics

The following definitions are introduced to simplify the notation throughout this section:

\[ \bar{x} \equiv \bar{x}(t), \bar{y} \equiv \bar{y}(t), \bar{W} \equiv \dot{W}(t), \bar{e} \equiv \varepsilon(x) \Omega \equiv \bar{x}^T(t) PB \]
\[ \beta \equiv \beta(x), \dot{\beta} \equiv \dot{\beta}(\hat{x}), \tilde{\beta} \equiv \beta(x) - \beta(\hat{x}) \]
\[ e \equiv e(t), \dot{e} \equiv \dot{e}(t), \dot{e}_m \equiv \dot{e}_m(t), \bar{W} \equiv \dot{W}(t) \]
\[ Q \equiv Q_0 + \mu NN^T \]

The next theorem states the main result.

Theorem 3.1. Consider the system comprised of the plant dynamics in (1) and the control law given in (7), made up of the nominal control law in (3) and the adaptive control law in (8), together with the state observer in (9) and the weight update law in (13) with \( \mu < \bar{\mu} \). Under Assumptions 2.1 and 2.2, for a sufficiently large \( D \), and if \( \lambda_{\min}(Q_0) > 2\|PB\|\omega^T L_{\beta} \), then \( \bar{x}(t) \) and \( \bar{W}(t) \) are UUB.

Proof: Consider the Lyapunov candidate given by

\[ V(\bar{x}, \bar{W}) = \bar{x}^T P \bar{x} + \gamma^{-1} \text{tr} \left( \bar{W}^T \bar{W} \right) \]

where \( P \) is a positive definite solution of (12) for \( 0 < \mu < \bar{\mu} \). The time derivative of (27) along closed loop solutions of (1) is given by

\[ \dot{V} = -\bar{x}^T Q \bar{x} + 2 \bar{x}^T PB \bar{\Delta} + 2 \gamma^{-1} \text{tr} \left( \bar{W}^T \dot{\bar{W}} \right) \]

where \( \dot{V} \equiv \dot{V}(\bar{x}, \bar{W}) \). With the proposed weight update law in (13) and \( \tilde{\Delta}(t) \) in (24), (28) can be written as
\[ V = -\ddot{x}^T Q \ddot{x} + 2\Omega [W^T \hat{\beta} + W^T \hat{\beta} + \epsilon] \\
- 2 \text{tr} \left[ W^T \hat{\beta} \ddot{y}^T - \sigma W^T \dot{W} - W^T \hat{\beta} \dot{W} / 2\mu \right] \\
= -\ddot{x}^T Q \ddot{x} + 2\Omega [W^T \hat{\beta} + W^T \hat{\beta} + \epsilon] \\
- 2 \text{tr} \left[ W^T \hat{\beta} \ddot{y}^T - \sigma W^T \dot{W} + \sigma W^T \dot{W} - W^T \hat{\beta} \dot{W} / 2\mu + W^T \hat{\beta} \dot{W} / 2\mu \right] \] (29)

Using the expression for \( N \) in (12), the definition of \( \Omega \) in (26), and the trace property \( a^T b = \text{tr}[ba^T] \) for \( a, b \in \mathbb{R}^m \), (29) becomes
\[
V = -\ddot{x}^T Q \ddot{x} + 2\Omega [W^T \hat{\beta} + \epsilon] \\
\quad - 2\ddot{x}^T N W^T \hat{\beta} - 2\sigma \text{tr} [W^T \dot{W}] + 2\sigma \text{tr} [W^T \dot{W}] + \hat{\beta}^T W \dot{W} \hat{\beta} / \mu - \hat{\beta}^T W \dot{W} \hat{\beta} / \mu \] (30)

From Young’s inequality,\(^{22}\) we have that \( 2a^T b \leq \nu a + b^T b / \nu \), \( \nu > 0 \) for any vectors \( a \) and \( b \). This can be generalized to matrices as \( 2 \text{tr}[A^T B] \leq \nu \text{tr}[A^T A] + \text{tr}[B^T B] / \nu \), for any matrices \( A \) and \( B \) with compatible dimensions. Applying the vector form to the third term in (30) and the matrix form with \( \nu = 1 \) to the sixth term in (30), we obtain
\[
V \leq -\ddot{x}^T Q \ddot{x} + 2\Omega [W^T \hat{\beta} + \epsilon] + \mu \ddot{x}^T N N^T \ddot{x} + \frac{1}{\mu} \hat{\beta}^T W W^T \hat{\beta} - \sigma \text{tr} [W^T \dot{W}] + \sigma \text{tr} [W^T \dot{W}] \] (31)

Using Assumptions 2.1 and 2.2,
\[
2\Omega [W^T \hat{\beta} + \epsilon] \leq 2|\Omega| ||W|| F |x - \ddot{x}| + 2|\Omega| \|\epsilon\| \leq 2\|PB\| \|\omega^T L_\beta \| |x|^2 + 2\|PB\| \|\bar{x}\| |\ddot{x}| \] (32)

Also
\[
\frac{1}{\mu} \hat{\beta}^T W W^T \hat{\beta} \leq \frac{1}{\mu} \bar{B}^2 \|W\| F \|\ddot{W}\| F \] (33)

With (32) and (33), it follows from (31) that
\[
V \leq -\ddot{x}^T \left( Q - \mu NN^T \right) \ddot{x} + 2\|PB\| \|\omega^T L_\beta \| |x|^2 + 2\|PB\| \|\bar{x}\| \|\ddot{x}\| \\
\quad - \sigma \|\ddot{W}\| F \|\ddot{W}\| F + \sigma \|W\| F \|W\| F \] (34)

Substituting for \( Q \) from (26), (34) can be expressed as
\[
V \leq -\ddot{x}^T \left( Q_0 + 2\|PB\| \|\omega^T L_\beta \| |x|^2 + 2\|PB\| \|\bar{x}\| \|\ddot{x}\| \\
\quad - \sigma \|\ddot{W}\| F \|\ddot{W}\| F + \sigma \|W\| F \|W\| F \right) \] (35)

Using
\[
\lambda_{\min}(Q) \|\ddot{x}\|^2 \leq \ddot{x}^T Q \ddot{x} \leq \lambda_{\max}(Q) \|\ddot{x}\|^2 \] (36)

it follows from (35) that
\[
V \leq - \left( \lambda_{\min}(Q_0) - 2\|PB\| \|\omega^T L_\beta \| \right) \|\ddot{x}\|^2 + 2\|PB\| \|\bar{x}\| \|\ddot{x}\| - \sigma \|\ddot{W}\| F \|\ddot{W}\| F + \sigma \|W\| F \|W\| F \] (37)
Defining:
\[
\begin{align*}
    c & \equiv \lambda_{\min}(Q_0) - 2\|PB\|\omega^* L_\beta > 0 \\
    d_1 & \equiv \|PB\|\epsilon \\
    d_2 & \equiv \frac{1}{2\mu} \tilde{B}^2 \|W\|_F \\
    e^2 & \equiv \frac{d_1^2}{c} + \frac{d_2^2}{\sigma} + \sigma \|W\|_F^2 > 0
\end{align*}
\]  
(38)

(37) can be written as
\[
\mathcal{V} \leq -c\left(\frac{d_1}{c}\right)^2 - \sigma\left(\|W\|_F - \frac{d_2}{\sigma}\right)^2 + e^2
\]

Consequently, we can conclude that either of the following conditions:
\[
|x| > \Psi_1, \quad \|\tilde{W}\| > \Psi_2
\]
renders \(\mathcal{V}(\tilde{x}, \tilde{W}) < 0\), where \(\Psi_1 = e/\sqrt{c} + d_1/c\) and \(\Psi_2 = e/\sqrt{\sigma} + d_2/\sigma\), and it follows that \(\tilde{x}\) and \(\tilde{W}\) are UUB.  

**Remark 3.1.** From Lemma 2.1, the condition \(c > 0\) can be satisfied by choosing \(k\) sufficiently large.

**Corollary 3.1.** Under the conditions of Theorem 3.1, an estimate for the ultimate bound is given by
\[
r = \sqrt{\frac{\lambda_{\max}(P)\Psi_1^2 + \gamma^{-1}\Psi_2^2}{\lambda_{\min}(\tilde{P})}}
\]
(41)

where \(\tilde{P} = \text{diag}[P, \gamma^{-1}I]\).

**Proof:** Define \(\zeta = [\tilde{x}^T, \text{vec}(\tilde{W})]^T\) and denote the following sets:
\[
\begin{align*}
    B_r & = \{\zeta : |\zeta| \leq r\} \\
    \Omega_\alpha & = \{\zeta \in B_r : \zeta^T \tilde{P} \zeta \leq \alpha\}
\end{align*}
\]
(42)
where \(B_r \subset D_\zeta\) for a sufficiently large compact set \(D_\zeta\) and \(\alpha = \min_{|\zeta|=r}(\zeta^T \tilde{P} \zeta) = r^2 \lambda_{\min}(\tilde{P})\). A geometric description of these sets is given in Figure 2. From
\[
V(\tilde{x}, \tilde{W}) = \zeta^T \tilde{P} \zeta = \tilde{x}^T P \tilde{x} + \gamma^{-1} \text{tr}[\tilde{W}^T \tilde{W}]
\]
(43)
and it follows that \(\Omega_\alpha\) is an invariant set if and only if
\[
\alpha \geq \lambda_{\max}(P)\Psi_1^2 + \gamma^{-1}\Psi_2^2
\]
(44)
Therefore the minimum size of \(B_r\) is defined by (41).

**Remark 3.2.** The proofs of Theorem 3.1 and Corollary 3.1 assume the sets \(D\) and \(D_\zeta\) are sufficiently large. If we define \(B_R\) as the largest ball \(\subset D_\zeta\), and assume the initial conditions are such that \(\zeta(0) \subset B_R\), then from Figure 2 we have added the condition that \(r < R\), which implies an upper bound on \(\gamma\). It can be shown that in this case the upper bound must be such that \(\lambda_{\min}(\tilde{P}) = \gamma^{-1}\). With \(r\) defined by (41) and \(\lambda_{\min}(\tilde{P}) = \gamma^{-1}\), the condition \(r < R\) implies
\[
\gamma < \frac{R^2 - \Psi_2^2}{\lambda_{\max}(P)\Psi_1^2}
\]
(45)
Therefore, the meaning of ‘\(D_\zeta\) sufficiently large’ in Corollary 3.1 is that \(R > \sqrt{\gamma \lambda_{\max}(P)\Psi_1^2 + \Psi_2^2}\) and \(\zeta(0) \subset B_R\). The meaning of ‘\(D\) sufficiently large’ is difficult to characterize since \(x(t)\) depends on both \(r(t)\) and \(x(0)\).
Corollary 3.2. If \( \bar{x} \) is bounded, then the state tracking error \( e \) is bounded.

Proof: The state tracking error \( e \) can be expressed as a sum of the state estimation error \( \hat{e} \) and the estimated state tracking error \( \hat{e}_m \):

\[
|e| = |x - x_m| = |x - \hat{x} + \hat{x} - x_m| \\
\leq |x - \hat{x}| + |\hat{x} - x_m| = |\bar{x}| + |\hat{e}|
\]

From (11) it follows that if \( \bar{x} \) is bounded, then \( \hat{e} \) is bounded, and therefore \( e \) is bounded. \hfill \square

Corollary 3.3. Consider the system of equations in (11). If \( \bar{x} \) is UUB by \( r \) then \( e \) is UUB by \( r(1 + v) \), where

\[
v = \frac{2\|P_m LC\|}{\lambda_{\min}(Q_m)}
\]

Proof: Consider the positive semi-definite function

\[
V(\hat{e}, \bar{x}) = \hat{e}^T P_m \hat{e}
\]

From (11), there exists \( P_m > 0 \) for any \( Q_m > 0 \) such that

\[
0 = A_m^T P_m + P_m A_m + Q_m
\]

Then the time derivative of (48) along the trajectories of (11) can be written as

\[
V(\hat{e}, \bar{x}) = -\hat{e}^T Q_m \hat{e} + 2\hat{e}^T P_m LC \bar{x} \\
\leq -|\hat{e}| \left[ \lambda_{\min}(Q_m)|\bar{x}| - 2\|P_m LC\||\bar{x}| \right] \\
= -|\hat{e}| \frac{\lambda_{\min}(Q_m)}{\lambda_{\min}(Q_m)} \left[ |\bar{x}| - 2\|P_m LC\||\bar{x}| \right]
\]

From (11) it follows that \( \hat{e} \) is bounded so long as \( \bar{x} \) is bounded. It therefore follows that once \( \bar{x} \) reaches its ultimate bound that \( V(\hat{e}, \bar{x}) < 0 \) for all \( |\hat{e}| > rv \), and from (46) that \( |e| \) is UUB by \( r(1 + v) \). \hfill \square
IV. Illustrative Example

Wing-rock is a nonlinear phenomenon in which the aircraft undergoes limit cycle roll oscillations at high angles of attack. A simple two state model for wing-rock can be written in the form of (1), where \( x(t) = [\phi(t), \dot{\phi}(t)]^T \) represents roll angle (rad) and roll rate (rad/s); \( u(t) = \hat{\delta}_a \) is aileron deflection; and \( y(t) = \phi(t) \). The wing-rock dynamics are:

\[
\begin{bmatrix}
\dot{\phi}(t) \\
\dot{\phi}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\phi(t) \\
\dot{\phi}(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
1
\end{bmatrix}
u(t) + \Delta(x)
\]

(51)

where

\[
\Delta(x) = [b_0 + b_1 \phi(t) + b_2 \dot{\phi}(t) + b_3 \phi(t) \dot{\phi}(t) + b_4 \phi(t) | \dot{\phi}(t) + b_5 \phi(t)]
\]

(52)

where \( b_0 = 0, b_1 = -0.0186, b_2 = 0.0152, b_3 = 0.6245, b_4 = -0.0095, \) and \( b_5 = 0.0215 \) are aerodynamic coefficients. The control objective is to track a reference roll angle command. A bias term and five sigmoidal basis functions \( \beta_i(x) \) are used to form the basis vector, so \( \beta_1 = 1 \) and

\[
\beta_i(x) = \frac{1 - e^{-ax_i}}{1 + e^{-ax_i}}, \quad i = 2, 3, ..., 6
\]

with \( a = 3 \) and normalized \( x_i = [\phi_n, \dot{\phi}_n, \phi_n^2, \dot{\phi}_n^2, \phi_n \dot{\phi}_n, \dot{\phi}_n^3] \), where \( \phi_n = \hat{\phi} / \pi \) and \( \dot{\phi}_n = \dot{\phi} / \pi \).

A. Augmentation of a Proportional Nominal Controller

We will first illustrate a typical result when augmenting a nominal control design based on proportional control. The reference model is second order with a natural frequency of 1.5 rad/sec and a damping ratio of 0.707. This amounts to choosing \( K_r = [2.25, 2.121] \) and \( K_r = 2.25 \). The observer gain \( L = [7.07, 25]^T \) so that \( \lambda(A_r) = 4\lambda(A_m) \) (\( k = 4 \)).

Figure 3 shows the \( \bar{\mu} \) boundary versus \( k \) for \( Q_0 = I_2 \). For \( k = 4 \), \( \bar{\mu} = 23.8 \). Figure 4 shows the ultimate bound for \( \bar{x} \) and \( \bar{W} \) versus \( \mu \) for the case \( k = 4 \) in the observer design, \( \sigma = 0.01 \), and assuming \( \omega^* = 0.1 \). Note that the ultimate bounds are minimized by choosing \( \mu \) close to \( \bar{\mu} \). Figure 5 confirms that \( c > 0 \) for the complete range for \( \mu \), for \( k = 4 \).

Simulation results are given in Figures 6 - 9 for \( k = 4 \) in the observer design. These were generated using \( \gamma = 100 \), \( \sigma = 0.01 \), \( \mu = 23.7 \), and randomly selected initial conditions \( \phi(0) = 6^\circ \) and \( \dot{\phi}(0) = 3^\circ/\text{sec} \). Figure 6 shows a typical step response. Note that the system is unstable without adaptation, but it tracks the reference response reasonably well with adaptation. However there is a significant steady state error. Figure 7 shows the response for a sequence of pulse commands. The comparison between \( u_{ad}(t) \) and \( \Delta(x) \) is given in Figure 8 and the corresponding weight histories are shown in Figure 9.

B. Augmentation of a PI Nominal Controller

The previous results highlight the fact that we cannot rely on the adaptive controller to provide zero steady state error when \( \Delta(x) \neq 0 \). If zero steady state error in the presence of constant uncertainty or constant disturbances is a requirement, then this requirement should be reflected in the design of the nominal control law. This point illustrated in the remainder of this section.

A third state variable is added to the dynamics that represents the integral of \( r(t) - \phi(t) \). The new system becomes

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B[u(t) + \Delta(x(t))] + B r(t) \\
y(t) &= C x(t)
\end{align*}
\]

(54)

where \( x(t) = [\phi(t), \dot{\phi}(t), f(r(t) - \phi(t))]^T \) and
The feedback gain matrix in the nominal controller was designed as a linear quadratic regulator (LQR). The weighting matrices in the design were chosen as $Q = \text{diag}[20, 3, 1]$ and $R = 0.5$. This leads to $K_r = [7.2675, 4.5316, -1.4142]$ as the PI feedback gain and $K_2 = 7.2675$ as the feedforward gain. In this case the form of the observer dynamics in (9) becomes

$$\dot{x}(t) = A\hat{x}(t) + Bu(t) + B_r r(t) + L[y(t) - \hat{y}(t)]$$

and the form of the error dynamics in (11) remains the same. Therefore the theorem and corollaries of the previous section still apply.

Figure 10 shows the $\tilde{\mu}$ boundary versus $k$ for $Q_0 = I_3$. The value of $\tilde{\mu}$ for $k = 4$ in this case is 71.2. It was again verified that the value of $\mu$ that minimizes the ultimate bounds for $\hat{x}$ and $\hat{W}$ is very close to this value. Step responses for the the case of PI nominal control design are given in Figures 11. These were generated using the same adaptation gains used previously, $\mu = 71$, and initial conditions $[6^\circ, 3^\circ/sec, 0]^T$. The system shows a significant oscillation without adaptation, but it tracks the reference response reasonably well with adaptation. Furthermore, the steady state error is zero.

V. Conclusion

This paper presents an architecture for output feedback model reference adaptive control that augments a nominal fixed gain controller. It is assumed that the nominal controller employs an observer. The observer is used in the adaptive part of the design in place of a reference model. The level of complexity for realization of proposed architecture, above that of the nominal controller, is far less than other methods. The error signals including state estimation error, estimated state tracking error, and weight estimate error are proven to be uniformly ultimately bounded. The stability analysis employs a Lyapunov candidate based on the existence of a positive definite solution of a parameter dependent Riccati equation. The upper limit for this parameter is employed in the design of the adaptive law, and also influences the ultimate bounds for the state estimate error and the weight error. A simulation example illustrates the computation of the parameter bound, computation of its influence on the ultimate error bounds, and a set of typical simulation results highlighting the attainable performance. Extensions of the proposed approach to K-modification, adaptive loop recovery, and derivative-free model reference adaptive control are ongoing research efforts.

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References


![Figure 3. Limit value of $\mu$ for $Q_0 = I_2$ using a proportional nominal controller.](image-url)
Figure 4. Ultimate bounds for $k = 4$.

Figure 5. $c$ for $k = 4$. 
Figure 6. Step responses with and without adaptation using a proportional nominal controller.

Figure 7. Tracking responses with and without adaptation.
Figure 8. $u_{ad}(t)$ vs $\Delta(x)$.

Figure 9. History of the estimated weight in tracking responses.
Figure 10. Limit value of $\mu$ for $Q_0 = I_3$ using a PI nominal controller.

Figure 11. Step responses with and without adaptation using a PI nominal controller.