Derivative-Free Output Feedback Adaptive Control

Tansel Yucelen,* Kilsoo Kim,† and Anthony J. Calise‡

Georgia Institute of Technology, Atlanta, GA, 30332-0150 USA

This paper presents an output feedback adaptive control architecture for continuous-time uncertain dynamical systems based on state observer and derivative-free delayed weight update law. The assumption of constant unknown ideal weights is generalized to the existence of time-varying weights without assuming the existence of their derivatives in a time interval. As a result, this approach is particularly well suited for adaptation in the presence of sudden change in uncertain system dynamics, such as might be due to reconfiguration, deployment of a payload, docking, structural damage, or there is a difficult to model disturbance. Using a Lyapunov-Krasovskii functional, it is proven that the error dynamics are uniformly ultimately bounded, without the need for modification terms in the adaptive law. The complexity of the proposed approach is less than many other output feedback adaptive control architectures available in the literature and it can be used to augment an existing state observer based linear controller.

I. Introduction

Research in adaptive control is motivated by the desire to achieve greater performance without sacrificing robustness to modeling error. Adaptive controllers can be classified as either full state feedback or output feedback. The assumption of full state feedback leads to computationally simpler adaptive controller algorithms in comparison to output feedback algorithms. Output feedback adaptive controllers, however, are required for applications in which it is impractical or impossible to sense the entire state of the process, such as active noise suppression, active control of flexible structures, fluid flow control systems, combustion control processes, and low cost or expendable unmanned aerial vehicle applications. Models for these applications vary from reasonably accurate low-frequency models, in the case of structural control problems, to less accurate low-order models in the case of active control of noise, vibrations, flows, and combustion processes.

There has been a number of results in the recent decades focused on output feedback adaptive controllers. For example, Esfandiari and Khalil[1] introduced a high gain observer for the reconstruction of the unavailable states. Krstic, Kanellakopoulos, and Kokotovic[2], and Marino and Tomei[3] proposed backstepping approaches for output feedback adaptive control, which are affine with respect to unknown parameters. In Ref. 4, Kim and Lewis suggested a neural network architecture for the observer. Output feedback adaptive control using a high gain observer and neural networks has also been proposed in Ref. 5 by Seshagiri and Khalil for nonlinear systems represented by input output models. Another method involving the design of adaptive observer using function approximators and backsteppting was given by Choi and Farrel[6] for a limited class of systems that can be transformed to output feedback form in which nonlinearities depend on measurements only.

*Graduate Research Assistant, Student Member AIAA, School of Aerospace Engineering, tansel@gatech.edu.
†Graduate Research Assistant, Student Member AIAA, School of Aerospace Engineering, kilsoo.kim@gatech.edu.
‡Professor, Fellow Member AIAA, School of Aerospace Engineering, anthony.calise@aerospace.gatech.edu.

Calise, Hovakimyan, and Idan[7] developed an output feedback adaptive control architecture for nonlinear uncertain dynamical systems without using a state observer but with a stable low-pass filter to satisfy a strictly positive real conditions. It requires that the input vector to the neural networks be composed of current and past input output data. In Ref. 8, Hovakimyan, Nardi, Calise, and Kim considered an output feedback adaptive controller with an error observer instead of a state observer. In this approach, only the relative degree of the regulated system needs to be known and the basis functions employed depend only the available input output history. Volyanskyy, Haddad, and Calise[9] introduced $\mathcal{Q}$-modification to output feedback adaptive control. This approach assumes a restrictive matching assumption. It is important to note that applying an adaptive control technique for uncertain dynamical systems in many cases implies replacement of an existing control system. On the other hand, it is highly desirable to consider an adaptive approach that can be implemented in a form of augmenting an existing controller. This rationale has been one of the viewpoints taken in applying adaptive control in Refs. 10–14.

Hoagg, Santillo, and Bernstein[15] estimate Markov parameters that leads to output feedback adaptive control algorithms with varying set of assumptions. Recently, Lavretsky[16] introduced an adaptive output feedback design using asymptotic properties of LQG/LTR controllers that asymptotically satisfies a strictly positive real (SPR) condition. Our approach is similar in spirit to that of Ref. 16, with the exception that it is less restrictive in that we do not attempt to satisfy an SPR condition, which naturally leads to a formulation that relies on the existence of a positive-definite solution of a parameter dependent Riccati equation. Therefore, our approach is applicable to nonminimum phase systems.

Adaptive controllers have in common the underlying assumption that there exists a constant, but unknown, ideal set of weights (see for example Refs. 1–16, and references therein). Although this assumption seems reasonable and these adaptive controllers work well on many systems, in some failure modes they may require the use of unrealistically high adaptation gain or may fail to achieve the desired level of performance in terms of failure recovery. Adaptive controllers that require high gain can excite unmodeled dynamics, typically exhibit an excessive amount of control activity[17,18], amplify the effect of sensor noise, and increase sensitivity to time delays[19].

In Refs. 20 and 21, Yucelen and Calise developed a derivative-free adaptive control (DF-AC) law. They generalized the assumption of constant unknown ideal weights to the existence of time-varying weights, such that fast and possibly discontinuous variation in weights are allowed. This generalization adds a dimension in the tuning process such that the adaptive law uses the information of delayed weight estimates and the information of the current known system states and errors. By providing memory to the adaptive law, this approach can be used to improve transient behavior without increasing the effective adaptation gain. In Ref. 22, DF-AC was applied to NASA Generic Transport Model (GTM)[23] and shown to provide a superior performance in comparison to a conventional adaptive law, when it is evaluated for a variety of cases in which there is a sudden or rapidly varying set of dynamics. The DF-AC law has also been successfully flight tested on NASA Airborne Subscale Transport Aircraft Research (AirSTAR) flight test vehicle, and the results are reported in Refs. 24 and 25.

This paper extends the results in Refs. 20 and 21 to output feedback adaptive control for continuous-time uncertain dynamical systems using a state observer. The observer is employed in the adaptive part of the design in place of a reference model. A derivative-free weight update law in output feedback form ensures that the estimated states follow both the reference model states and the true states so that both the state estimation error and the state tracking error are bounded. The proposed approach is particularly advantageous for applications to systems that can undergo a sudden change in dynamics, such as might be due to reconfiguration, deployment of a payload, docking, structural damage, or there is a difficult to model disturbance.

The stability analysis uses a Lyapunov-Krasovskii functional that entails the solution of a parameter dependent Riccati equation[26], rather than a Lyapunov equation, to show that all the error signals are uniformly ultimately bounded (UUB). It is shown how the Riccati equation parameter influences the ultimate bounds for the state estimate error and the adapted weight error. The complexity of the approach is significantly less than many other approaches to output feedback adaptive control and it can be implemented in a form that augments an existing observer based linear control architecture. The design procedure is illustrated on a model of wing rock dynamics.

The notation used in this paper is fairly standard. $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $(\cdot)^{\mathrm{T}}$ denotes transpose, $(\cdot)^{-1}$ denotes inverse, and $\otimes$ denotes a Kronecker product. We write $\underline{\lambda}(M)$ (resp., $\overline{\lambda}(M)$) for the minimum (resp., maximum) eigenvalue of matrix M, $|\cdot|$ for the Euclidean vector norm, $||\cdot||$ for the Euclidean matrix norm, $\mathrm{vec}(\cdot)$ for the column stacking operator, $\mathrm{tr}(\cdot)$ for the trace operator, and $\mathrm{diag}[M, N]$ for a block diagonal matrix formed with matrices M and N on the diagonal.

The organization of the paper is as follows. Section II presents the observer based derivative-free output feedback adaptive control architecture. Section III discusses the properties of a parameter dependent Riccati equation. Section IV provides the stability analysis for the architecture proposed in Section 2 based on the parameter dependent Riccati equation described in Section III. The error signals are shown to be UUB. Section V illustrates the proposed adaptive control approach on a model of wing rock dynamics. Finally, section VI summarizes the conclusions.

II. Derivative-Free Output Feedback Adaptive Control Architecture

Consider the uncertain time-varying dynamical system given by

$$\dot{x}(t) = Ax(t) + B\big[u(t) + \Delta(t, x(t))\big] \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times n}$ are known system matrices, $x(t) \in \mathbb{R}^n$ is the unknown state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $\Delta : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is the matched uncertainty, and $y(t) \in \mathbb{R}^m$ is the *regulated* output vector. Furthermore, the triple (A, B, C) is minimal and the control input vector is restricted to the class of admissible controls consisting of measurable functions.

Remark 2.1. The system given by (1) and (2) assumes that the control input vector and the regulated output vector have the same dimension. For the case when the dimension of the control input vector is larger than the dimension of the regulated output vector and there redundancy in actuation, one can use either matrix inverse and pseudoinverse approaches, constrained control allocation, pseudocontrols, or daisy chaining[27–29] to reduce the dimension of the control input vector to the dimension of the regulated output vector. Furthermore, the system can have a *sensed* output vector denoted by

$$y_{\mathrm{s}}(t) = C_{\mathrm{s}}x(t) \tag{3}$$

where $y_{\mathrm{s}}(t) \in \mathbb{R}^l$, $C_{\mathrm{s}} \in \mathbb{R}^{l\times n}$, $l \geq m$, such the elements of $y(t)$ are a subset of the elements of $y_{\mathrm{s}}(t)$.

Consider the state observer based nominal control law given by

$$u_{\mathrm{n}}(t) = -K_1\hat{x}(t) + K_2 r(t) \tag{4}$$

where $K_1 \in \mathbb{R}^{m\times n}$ and $K_2 \in \mathbb{R}^{m\times m}$ are known feedback and feedforward matrices, respectively, $r(t) \in \mathbb{R}^r$ is the regulated output command vector, and $\hat{x}(t) \in \mathbb{R}^n$ is an observer estimate of $x(t)$ given by:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu_{\mathrm{n}}(t) + L\big[y_{\mathrm{s}}(t) - C_{\mathrm{s}}\hat{x}(t)\big] \tag{5}$$

with $L \in \mathbb{R}^{n\times l}$ being the observer gain matrix designed such that $A_e \equiv A - LC_{\mathrm{s}} \in \mathbb{R}^{n\times n}$ is Hurwitz. Define the reference model:

$$\dot{x}_{\rm m}(t) = A_{\rm m}x_{\rm m}(t) + B_{\rm m}r(t) \tag{6}$$

where $x_{\rm m}(t) \in \mathbb{R}^n$, $A_{\rm m} = A - BK_1$ is Hurwitz by design, and $B_{\rm m} = BK_2$.

Remark 2.2. The above implies that the gains K_1 and K_2 have been designed using the certainty equivalence principle ($\hat{x}(t) = x(t)$) with $\Delta(t, x(t)) = 0$, so that $C\hat{x}(t)$ tracks $r(t)$ to within some set of specifications on both the transient and steady state performance.

Assumption 2.1. The matched uncertainty in (1) can be linearly parameterized as

$$\Delta(t, x(t)) = W^{\rm T}(t)\beta(x(t)), \quad \forall x \in \mathcal{D}_x \subset \mathbb{R}^n \tag{7}$$

where $W(t) \in \mathbb{R}^{s\times m}$ is the unknown time-varying weight matrix that satisfies $\|W(t)\| \leq \bar{\omega}$, $\beta : \mathbb{R}^n \to \mathbb{R}^s$ is a known Lipschitz continuous basis vector of the form $\beta(x) = [\beta_1(x), \beta_2(x), ..., \beta_s(x)]^{\rm T}$ with $|\beta(x)| \leq \bar{\beta}$, and $\mathcal{D}_x$ is sufficiently large.

Remark 2.3. Assumption 2.1 expands the class of uncertainties that can be represented by a given set of basis functions. That is, an adaptive law design subject to Assumption 2.1 can be more effective than an adaptive law designed subject to

$$\Delta(x) = W^{\rm T}\beta(x) + \varepsilon(x), \quad \forall x \in \mathcal{D}_x \tag{8}$$

where W is a unknown constant ideal weight and $\varepsilon(x)$ is the residual error. It also permits an explicit dependence of the uncertainty on time.

Remark 2.4. Assumption 2.1 does not place any restriction on the time derivative of the weight matrix. However the degree of time dependence will depend on how $\beta(x)$ is chosen.

The adaptive control objective is to design a control law $u(\cdot)$ for the system given by (1) and (2) so that $x(t)$ tracks $x_{\rm m}(t)$ with bounded error. For this purpose, the nominal control law $u_{\rm n}(t)$ given by (4) is augmented with the adaptive control law $u_{\rm ad}(t)$ as:

$$u(t) = u_{\rm n}(t) - u_{\rm ad}(t) \tag{9}$$

$$u_{\rm ad}(t) = \hat{W}^{\rm T}(t)\beta(\hat{x}(t)) \tag{10}$$

Note that the state observer given by (5) is regarded as a part of the nominal control design. However, our viewpoint below is that L may be altered for purposes of adaptively augmenting the nominal controller.

Denote $\tilde{x}(t) \equiv x(t) - \hat{x}(t)$ for the state estimation error, $\hat{e}(t) \equiv \hat{x}(t) - x_{\rm m}(t)$ for the estimated state tracking error, and $\tilde{W}(t) \equiv W(t) - \hat{W}(t)$ for the weight estimate error. From (1) and (5), the dynamics for the state estimation error can be written in the form

$$\dot{\tilde{x}}(t) = A_e\tilde{x}(t) + B\tilde{W}^{\rm T}(t)\beta(\hat{x}(t)) + Bg(x(t), \hat{x}(t)) \tag{11}$$

where $g(x(t), \hat{x}(t)) \equiv W^{\rm T}(t)\big[\beta(\hat{x}(t)) - \beta(x(t))\big]$ with $|g(x(t), \hat{x}(t))| \leq \bar{w}L_\beta|\tilde{x}(t)|$, where $L_\beta > 0$ is the Lipschitz constant for the known basis vector. Likewise, from (5) and (6) the dynamics for the estimated state tracking error can be written in the form:

$$\dot{\hat{e}}(t) = A_{\rm m}\hat{e}(t) + LC_{\rm s}\tilde{x}(t) \tag{12}$$

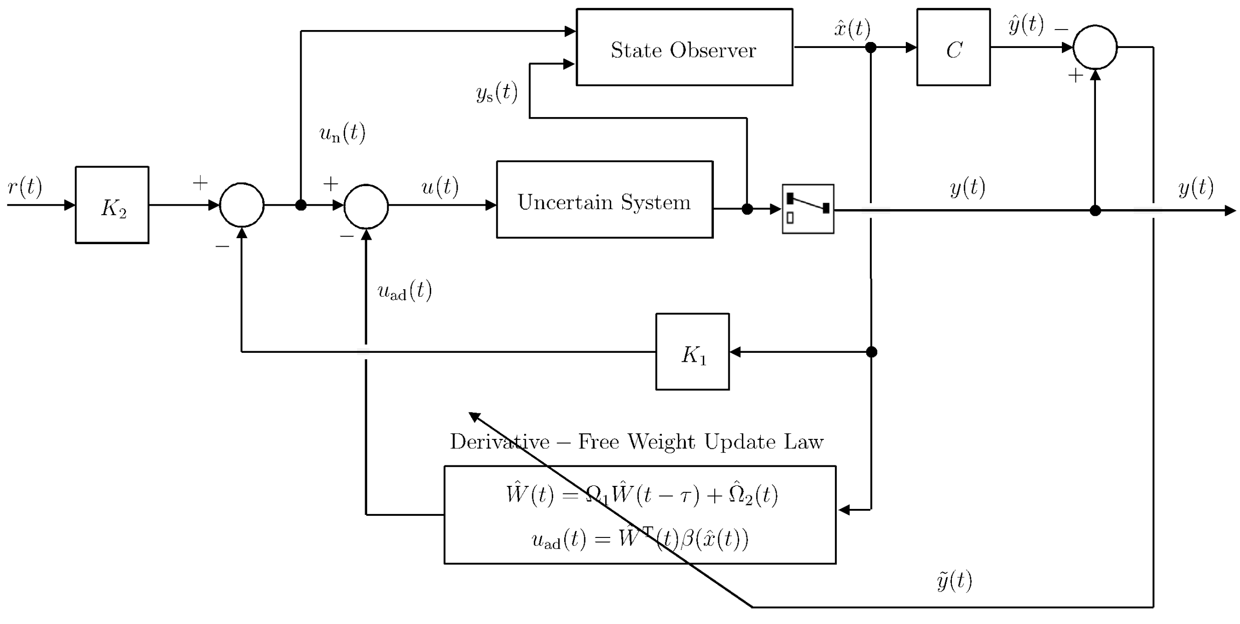

Figure 1. Derivative-free output feedback adaptive control architecture

Next, consider the derivative-free weight update law given by

$$\hat{W}(t) = \Omega_1\hat{W}(t-\tau) + \hat{\Omega}_2(t) \tag{13}$$

where $\tau > 0$, and $\Omega_1 \in \mathbb{R}^{s\times s}$ and $\hat{\Omega}_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{s\times m}$ satisfy:

$$0 \le \Omega_1^{\mathrm{T}}\Omega_1 < \kappa_1 I, \quad 0 \le \kappa_1 < 1/(1+\mu), \quad \mu > 0 \tag{14}$$

$$\hat{\Omega}_2(t) \equiv \kappa_2\beta(\hat{x}(t))\tilde{y}^{\mathrm{T}}(t), \quad \kappa_2 > 0 \tag{15}$$

where $\tilde{y}(t) \equiv y(t) - \hat{y}(t)$, where $\hat{y}(t) = C\hat{x}(t)$.

A visualization of the derivative-free output feedback adaptive control architecture is given in Figure 1. Note that the state observer serves as the reference model. Its dynamics are the same as the reference model if $u_{\mathrm{ad}}(t)$ cancels $\Delta(t, x(t))$, and in this case the observer error transient $\tilde{y}(t)$ goes to zero. Note that the components that are added to an observer based nominal controller architecture, in order to realize the adaptive control, reduce to computing the basis functions and implementing the adaptation law in (13).

III. A Parameter Dependent Riccati Equation

In this section we summarize the properties of a parameter dependent Riccati equation[26] that is needed for the stability analysis in Section IV. Consider the following quadratic equation in P

$$0 = A_e^{\mathrm{T}}P + PA_e + Q \tag{16}$$

$$Q = Q_0 + (\kappa_2 + 1/\mu)\bar{\beta}^2NN^{\mathrm{T}} = Q_0 + \nu NN^{\mathrm{T}} \tag{17}$$

$$N = C^{\mathrm{T}} - PB \tag{18}$$

in which $Q_0 \in \mathbb{R}^{n\times n} > 0$, and μ and $\bar{\beta}$ have previously been defined by (14) and Assumption 2.1, respectively.

Remark 3.1. Let $0 < \nu < \bar{\nu}$ define the largest set within which there exists a positive definite solution for P. Since $P > 0$ for $\nu = 0$ and P depends continuously on ν, the existence of $P(\nu) > 0$ for $0 < \nu < \bar{\nu}$ is assured. Furthermore, this implies that $\kappa_2 < \bar{\nu}/\bar{\beta}^2 - 1/\mu \equiv \bar{\kappa}_2$.

Remark 3.2. If $N = 0$ in (18), then it follows from (16)–(18) that we have

$$0 = A_e^{\mathrm{T}} P + P A_e + Q_0 \tag{19}$$
$$0 = C^{\mathrm{T}} - PB \tag{20}$$

which implies that the transfer function associated with the system $G(s) = C(sI - A_e)^{-1}B$ is positive-real[30]. In this case (16) reduces to a Lyapunov equation associated with the error dynamics in (11), which is commonly employed in the stability analysis of adaptive systems, and $\bar{\nu} = \infty$. This suggests that for the purposes of adaptive control design, when $m > 1$, it is advantageous to define a new measurement by taking a linear combination of the existing measurements

$$y_o(t) = My(t) = MCx(t) = C_o x(t) \tag{21}$$

where $M \in \mathbb{R}^{m \times m}$ is a norm preserving transformation that minimizes a norm measure of N_o where

$$N_o \equiv C_o^{\mathrm{T}} - P_o B \tag{22}$$

with P_o defined as the value of P that satisfies (19). Taking the Frobenius norm as a measure, it can be shown that the solution for M that minimizes $||N_o||_{\mathrm{F}}$ subject to the constraint $||MC||_{\mathrm{F}} = ||C||_{\mathrm{F}}$ is given by:

$$M = kM_o \tag{23}$$
$$M_o^{\mathrm{T}} = \left[CC^{\mathrm{T}}\right]^{-1} C P_o B \tag{24}$$
$$k = \frac{||C||_{\mathrm{F}}}{||M_o C||_{\mathrm{F}}} \tag{25}$$

Lemma 3.1. If A_{m} has no repeated eigenvalues and the state observer gain matrix L of (5) is designed using pole placement such that

$$\lambda(A_e) = k\lambda(A_{\mathrm{m}}) \tag{26}$$

then for $\nu < \bar{\nu}$, we can denote $P(k)$ as the corresponding positive definite solution for P of the parameter dependent Riccati equation given by (16)–(18). In this case:

$$\lim_{k \to \infty} P(k) = 0 \tag{27}$$

Proof. Let S_{m} and S_e be diagonalizing transformations for A_{m} and A_e, respectively. Then:

$$A_{\mathrm{m}} = S_{\mathrm{m}} \mathrm{diag}[\lambda(A_{\mathrm{m}})] S_{\mathrm{m}}^{-1} \tag{28}$$
$$A_e = k S_e \mathrm{diag}[\lambda(A_{\mathrm{m}})] S_e^{-1} \tag{29}$$

With $S_{\mathrm{s}} \equiv S_e S_{\mathrm{m}}^{-1}$, A_e can be written as:

$$A_e = k S_{\mathrm{s}} A_{\mathrm{m}} S_{\mathrm{s}}^{-1} = k A_{\mathrm{s}} \tag{30}$$

Using (30) and denoting $P_k \equiv kP(k)$ and $N_k \equiv C^{\mathrm{T}} - k^{-1} P_k B$, the parameter dependent Riccati equation in (16)–(18) can be written as:

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P_k + P_k A_{\mathrm{s}} + Q_0 + \nu N_k N_k^{\mathrm{T}} \tag{31}$$

Taking the limit of (31) as $k \to \infty$, it follows that:

$$0 = A_{\mathrm{s}}^{\mathrm{T}} P_\infty + P_\infty A_{\mathrm{s}} + Q_0 + \nu C C^{\mathrm{T}} \tag{32}$$

Since the solution of (32) is finite, it follows that for all finite k, $P(k) > 0$, and:

$$\lim_{k\to\infty} P(k) = \lim_{k\to\infty} \frac{P_k}{k} = 0 \tag{33}$$

□

Remark 3.3. Lemma 3.1 implies that $||PB||$ can be made arbitrarily small by making the observer dynamics sufficiently fast, while preserving the property that $P > 0$. This property will become important in the stability analysis of Section IV.

The next lemma shows that for $\nu < \bar{\nu}$, (16)–(18) can reliably be solved for $P > 0$ using the Potter approach given in Ref. 31. This also implies that $\bar{\nu}$ can be determined by searching for the boundary value that results in a failure of the algorithm to converge. We employ the notation ric$(\cdot)$ and dom(ric) as defined in Ref. 32.

Lemma 3.2. Let P satisfy the parameter dependent Riccati equation given by (16)–(18) and let the modified Hamiltonian be given by

$$H \equiv \begin{bmatrix} A_e - \nu BC & \nu R \\ -Q & -\big(A_e - \nu BC\big)^{\mathrm{T}} \end{bmatrix} \tag{34}$$

where $Q \equiv Q_0 + \nu CC^{\mathrm{T}}$ and $R \equiv BB^{\mathrm{T}}$. Then, for all $0 < \nu < \bar{\nu}$, $H \in \mathrm{dom(ric)}$ and $P = \mathrm{ric}(H)$.

Proof: The proof follows from Lemmas 1 and 2 of Ref. 32. □

IV. Stability Analysis

This section presents a stability analysis for the derivative-free output feedback adaptive control architecture in Section II.

Theorem 4.1. Consider the controlled uncertain dynamical system given by (1) and (2) subject to Assumption 2.1. Consider, in addition, the feedback control law given by (9), with the nominal feedback control component given by (4) and (5), and the adaptive feedback control component given by (10) and (13) subject to the conditions in (14) and (15), and with $\kappa_2 < \bar{\kappa}_2$. Then, $\tilde{x}(t)$ and $\tilde{W}(t)$ are UUB.

Proof. Using (13) and defining

$$\Omega_2(t) \;\; \equiv \;\; W(t) - \Omega_1 W(t-\tau) \tag{35}$$

where $||\Omega_2(t)|| \leq \bar{\delta}$, $\bar{\delta} \equiv \bar{w}(1 + ||\Omega_1||)$, the weight update error $\tilde{W}(t)$ can be rewritten as:

$$\tilde{W}(t) \;\; = \;\; \Omega_1 \tilde{W}(t-\tau) + \Omega_2(t) - \hat{\Omega}_2(t) \tag{36}$$

Using (36), the state estimation error in (11) becomes:

$$\dot{\tilde{x}}(t) \;\; = \;\; A_e \tilde{x}(t) + B\big[\Omega_1 \tilde{W}(t-\tau) + \Omega_2(t) - \hat{\Omega}_2(t)\big]^{\mathrm{T}} \beta(\hat{x}(t)) + Bg(x(t), \hat{x}(t)) \tag{37}$$

Note that $\hat{\Omega}_2(t)$, given by (15), can be equivalently written using (18) as:

$$\begin{aligned} \hat{\Omega}_2(t) \;\; &= \;\; \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) C^{\mathrm{T}} \\ &= \;\; \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) \big[PB + N\big] \\ &= \;\; \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) PB + \kappa_2 \beta(\hat{x}(t)) \tilde{x}^{\mathrm{T}}(t) N \end{aligned} \tag{38}$$

To show that the closed-loop system given by (36) and (37) is UUB, consider the Lyapunov-Krasovskii functional[33]

$$\mathcal{V}(\tilde{x}(t),\tilde{W}_t) = \tilde{x}^{\mathrm{T}}(t)P\tilde{x}(t) + \rho\ \mathrm{tr}\big[\int_{t-\tau}^{t} \tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\mathrm{d}s\big] \tag{39}$$

where $\rho > 0$, $\tilde{W}_t$ represents $\tilde{W}(t)$ over the time interval $t-\tau$ to t, and P satisfies (16) with $\nu < \bar{\nu}$, which implies $\kappa_2 < \bar{\kappa}_2$ as noted in Remark 3.1. The directional derivative of (39) along the closed-loop system trajectories of (36) and (37) is given by

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;=\;& -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) + 2\tilde{x}^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(\hat{x}(t))\\ & -2\tilde{x}^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(\hat{x}(t)) + 2\tilde{x}^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}\beta(\hat{x}(t))\\ & +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) + \eta\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) - \tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau)\big]\end{aligned} \tag{40}$$

where $\eta \equiv 1+\xi$, $\xi > 0$. Using (36) to expand the term $\mathrm{tr}[\eta\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)]$ in (40) produces:

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;=\;& -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) + 2\tilde{x}^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(\hat{x}(t))\\ & -2\tilde{x}^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(\hat{x}(t)) + 2\tilde{x}^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}\beta(\hat{x}(t))\\ & +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) - \tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau) + \eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\\ & +\eta\hat{\Omega}_2^{\mathrm{T}}(t)\hat{\Omega}_2(t) + \eta\Omega_2^{\mathrm{T}}\Omega_2 - 2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_1\tilde{W}(t-\tau)\\ & +2\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_2 - 2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_2\big]\end{aligned} \tag{41}$$

Next, consider the fact $a^{\mathrm{T}}b \le \gamma a^{\mathrm{T}}a + b^{\mathrm{T}}b/4\gamma$, $\gamma > 0$, that follows from Young's inequality[34,35] for any vectors a and b. This can be generalized to $\mathrm{tr}\big[A^{\mathrm{T}}B\big] = \mathrm{vec}(A)^{\mathrm{T}}\mathrm{vec}(B) \le \gamma\mathrm{vec}(A)^{\mathrm{T}}\mathrm{vec}(A) + \mathrm{vec}(B)^{\mathrm{T}}\mathrm{vec}(B)/4\gamma = \gamma\mathrm{tr}\big[A^{\mathrm{T}}A\big] + \mathrm{tr}\big[B^{\mathrm{T}}B\big]/4\gamma$, $\gamma > 0$, for any matrices A and B with appropriate dimensions. Using this, we can write:

$$\mathrm{tr}\big[2\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_2(t)\big] \;\le\; \mathrm{tr}\big[\gamma\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\big] + \mathrm{tr}\big[\eta^2\Omega_2^{\mathrm{T}}(t)\Omega_2(t)/\gamma\big], \quad \gamma > 0 \tag{42}$$

Letting $\kappa_2 = 1/\rho\eta$, using (38), and substituting (42) to (41), it can be shown after tedious algebraic steps (see Appendix A) that

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;\le\;& -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) - \kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t))\\ & -\rho\xi\ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] - \rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big]\\ & +\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2] - \rho\kappa_2\ \mathrm{tr}[2\eta N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_2]\\ & +\kappa_2\tilde{x}^{\mathrm{T}}(t)NN^{\mathrm{T}}\tilde{x}(t)\bar{\beta}^2 - 2\ \mathrm{tr}[N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_1\tilde{W}(t-\tau)]\end{aligned} \tag{43}$$

where using Young's inequality again for the last term in (43) produces:

$$\begin{aligned}2\ \mathrm{tr}[N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_1\tilde{W}(t-\tau)] \;\le\;& \mathrm{tr}[\mu\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)]\\ & +\tilde{x}^{\mathrm{T}}(t)NN^{\mathrm{T}}\tilde{x}(t)\bar{\beta}^2/\mu, \quad \mu > 0\end{aligned} \tag{44}$$

Using (44) in (43) results in

$$\begin{aligned}\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \;\le\;& -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) - \kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t))\\ & -\rho\xi\ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] - \rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma+\mu)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big]\\ & +\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2] - \rho\kappa_2\ \mathrm{tr}[2\eta N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_2]\\ & \big(\kappa_2+1/\mu\big)\bar{\beta}^2\tilde{x}^{\mathrm{T}}(t)NN^{\mathrm{T}}\tilde{x}(t)\end{aligned} \tag{45}$$

Applying the definitions of Q and ν in (17), and enforcing $\kappa_2 < \bar{\kappa}_2$, the first and last terms in (45) can be combined to produce the following inequality

$$
\begin{aligned}
\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \quad \le \quad & -\tilde{x}^{\mathrm{T}}(t)Q_0\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) - \kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t)) \\
& -\rho\xi\ \mathrm{tr}[\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)] - \rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma+\mu)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big] \\
& +\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2] - \rho\kappa_2\ \mathrm{tr}[2\eta N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\Omega_2] \qquad (46)
\end{aligned}
$$

Define $\kappa_1 \equiv 1/(\eta+\gamma+\mu)$. Note that $\mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big] = \mathrm{vec}\big(\tilde{W}(t-\tau)\big)^{\mathrm{T}}\mathrm{vec}\big([I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big) = \mathrm{vec}\big(\tilde{W}(t-\tau)\big)^{\mathrm{T}}\big(I\otimes[I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\big)\mathrm{vec}\big(\tilde{W}(t-\tau)\big) \ge \underline{\lambda}\big(I\otimes[I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1]\big)||\tilde{W}(t-\tau)||^2 = \underline{\lambda}(I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1)||\tilde{W}(t-\tau)||^2$. Using this and (17) in (46) with $|g(x(t),\hat{x}(t))| \le \bar{g}$, $\bar{g} \equiv 2\bar{w}\bar{\beta}$, yields

$$
\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \quad \le \quad -c_1|\tilde{x}(t)|^2 - c_2||\tilde{W}(t)||^2 - c_3||\tilde{W}(t-\tau)||^2 + d_1 + d_2|\tilde{x}(t)| \qquad (47)
$$

where $c_1 \equiv \underline{\lambda}(Q_0) > 0$, $c_2 \equiv \rho\xi > 0$, $c_3 \equiv \rho\underline{\lambda}(I-\kappa_1^{-1}\Omega_1^{\mathrm{T}}\Omega_1) > 0$, $d_1 \equiv \rho(\eta+\eta^2/\gamma)\bar{\delta}^2 \ge 0$, and $d_2 \equiv 2(||PB||\bar{g} + ||N||||\Omega_2||\bar{\beta}) > 0$. We can further arrange (47) as

$$
\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \quad \le \quad -\big[\sqrt{c_1}|\tilde{x}(t)| - d_2/2\sqrt{c_1}\big]^2 - c_2||\tilde{W}(t)||^2 - c_3||\tilde{W}(t-\tau)||^2 + d \qquad (48)
$$

where $d \equiv d_1 + d_2^2/4c_1$. Either $|\tilde{x}(t)| > \Psi_1$ or $\|\tilde{W}(t)\| > \Psi_2$ or $\|\tilde{W}(t-\tau)\| > \Psi_3$ renders $\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) < 0$, where $\Psi_1 \equiv \sqrt{d/c_1} + d_2/2c_1$, $\Psi_2 \equiv \sqrt{d/c_2}$, and $\Psi_3 \equiv \sqrt{d/c_3}$. Hence, it follows that $\tilde{x}(t)$ and $\tilde{W}(t)$ are UUB. □

Remark 4.1. The derivative-free weight update law given by (13) does not require a modification term to prove the error dynamics, including the weight errors, are UUB.

Remark 4.2. Derivative-free adaptive control does not employ an integrator in its weight update law. This is advantageous from the perspective of augmenting a nominal controller that employs integral action to ensure that the regulated output variables track $r(t)$ for constant disturbances, regardless of how these disturbances may enter the system. An example that illustrates this advantage is provided for full-state feedback case in Section V of Ref. 20.

Define $q(t) \equiv [\tilde{x}^{\mathrm{T}}(t), \tilde{v}(t,\tau)]^{\mathrm{T}}$, where $\tilde{v}^2(t,\tau) \equiv \mathrm{tr}\big[\int_{-\tau}^{0}\tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\mathrm{d}s\big]$, and let $\mathcal{B}_r = \{q(t) : |q(t)| < r\}$, such that $\mathcal{B}_r \subset \mathcal{D}_q$ for a sufficiently large compact set $\mathcal{D}_q$. Then, we have the following corollary.

Corollary 4.1. Under the conditions of Theorem 4.1, an estimate for the ultimate bound for $q(t)$ is given by

$$
r = \sqrt{\frac{\overline{\lambda}(P)\Psi_1^2 + \rho\tau\Psi_2^2}{\underline{\lambda}(\tilde{P})}} \qquad (49)
$$

where $\tilde{P} = \mathrm{diag}[P,\rho]$.

Proof. Denote $\Omega_\alpha = \{q(t) \in \mathcal{B}_r : q^{\mathrm{T}}(t)\tilde{P}q(t) \le \hat{\alpha}\}$, $\hat{\alpha} = \min_{||q(t)||=r} q^{\mathrm{T}}(t)\tilde{P}q(t) = r^2\underline{\lambda}(\tilde{P})$. Since

$$
\begin{aligned}
\mathcal{V}(\tilde{x}(t),\tilde{W}_t) \quad &= \quad q^{\mathrm{T}}(t)\tilde{P}q(t) \\
&= \quad \tilde{x}^{\mathrm{T}}(t)P\tilde{x}(t) + \rho\ \mathrm{tr}\Big[\int_{t-\tau}^{t}\tilde{W}^{\mathrm{T}}(s)\tilde{W}(s)\mathrm{d}s\Big] \qquad (50)
\end{aligned}
$$

it follows that Ω_α is an invariant set if and only if:

$$
\hat{\alpha} \ge \overline{\lambda}(P)\Psi_1^2 + \rho\tau\Psi_2^2 \qquad (51)
$$

Thus, the minimum size of $\mathcal{B}_r$ that ensures this condition has radius given by (49). The sets used in this proof are illustrated in Figure 2. □

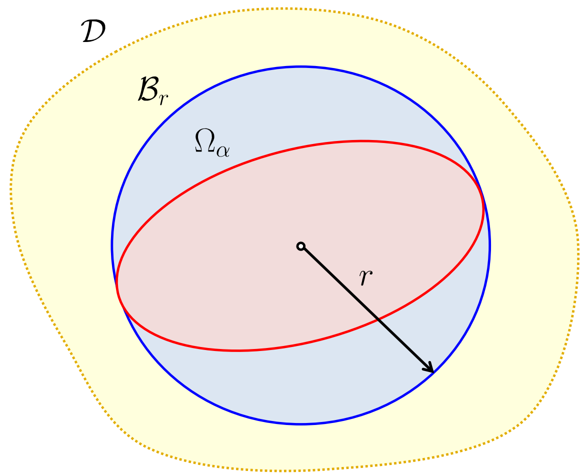

Figure 2. Geometric representation of sets.

Remark 4.3. The proofs of Theorem 4.1 and Corollary 4.1 assume that the sets $\mathcal{D}_x$ and $\mathcal{D}_q$ are sufficiently large. If we define $\mathcal{B}_{r^*}$ as the largest ball contained in $\mathcal{D}_q$, and assume that the initial conditions are such that $q(0) \subset \mathcal{B}_{r^*}$, then from Figure 2 we have added the condition that $r < r^*$, which implies an lower bound on ρ. It can be shown that in this case the lower bound must be such that $\underline{\lambda}(\tilde{P}) = \rho$. With r defined by (49) and $\underline{\lambda}(\tilde{P}) = \rho$, the condition $r < r^*$ implies:

$$\rho \quad > \quad \frac{\overline{\lambda}(P)\Psi_1^2}{r^{*2} - \tau\Psi_2^2} \tag{52}$$

Since $\kappa_2 = 1/\rho\eta$, $\eta > 1$, it follows from (52) that r^* should ensure that:

$$\kappa_2 \quad < \quad \frac{r^{*2} - \tau\Psi_2^2}{\overline{\lambda}(P)\Psi_1^2} \tag{53}$$

Therefore, the meaning of $\mathcal{D}_q$ *sufficiently large* in Corollary 4.1 is that $r^* > \sqrt{\kappa_2\overline{\lambda}(P)\Psi_1^2 + \tau\Psi_2^2}$ and $q(0) \subset \mathcal{D}_{r^*}$. The meaning of $\mathcal{D}_x$ *sufficiently large* is difficult to characterize precisely since $x(t)$ depends on both $r(t)$ and $x(0)$. Nevertheless it can be seen that increasing κ_2 implies increasing the require size of the set $\mathcal{D}_x$.

Lemma 4.1. If $\tilde{x}(t)$ is bounded, then the state tracking error defined as $e(t) \equiv x(t) - x_{\mathrm{m}}(t)$ is bounded.

Proof.

$$\begin{aligned} |e(t)| &= |x(t) - x_{\mathrm{m}}(t)| \\ &= |x(t) - \hat{x}(t) + \hat{x}(t) - x_{\mathrm{m}}(t)| \\ &\leq |x(t) - \hat{x}(t)| + |\hat{x}(t) - x_{\mathrm{m}}(t)| \\ &= |\tilde{x}(t)| + |\hat{e}(t)| \end{aligned} \tag{54}$$

where if $\tilde{x}(t)$ is bounded then from (12) $\hat{e}(t)$ is bounded. This implies that $e(t)$ is bounded. □

Corollary 4.2. Consider the system of equations given by (11) and (12). If $\tilde{x}(t)$ is UUB by r, then $e(t)$ is UUB by $r(1 + v)$ where

$$v \quad \equiv \quad \frac{2||P_{\mathrm{m}}LC_{\mathrm{s}}||}{\underline{\lambda}(Q_{\mathrm{m}})} \tag{55}$$

and $P_{\mathrm{m}} \in \mathbb{R}^{n \times n}$ being a positive-definite solution to the Lyapunov equation given by

$$0 = A_{\mathrm{m}}^{\mathrm{T}} P_{\mathrm{m}} + P_{\mathrm{m}} A_{\mathrm{m}} + Q_{\mathrm{m}} \tag{56}$$

for a given positive-definite matrix $Q_{\mathrm{m}} \in \mathbb{R}^{n \times n}$.

Proof. To show that $\hat{e}(t)$ is UUB, consider the Lyapunov function given by

$$\mathcal{V}(\hat{e}(t)) = \hat{e}^{\mathrm{T}}(t) P_{\mathrm{m}} \hat{e}(t) \tag{57}$$

where P_{m} satisfies (56). We can write the directional derivative of (57) along the trajectories of $\hat{e}(t)$ given by (12) as

$$\begin{aligned} \dot{\mathcal{V}}(\hat{e}(t)) &= -\hat{e}^{\mathrm{T}}(t) Q_{\mathrm{m}} \hat{e}(t) + 2\hat{e}^{\mathrm{T}}(t) P_{\mathrm{m}} L C_{\mathrm{s}} \tilde{x}(t) \\ &\leq -|\hat{e}(t)| \Big[\underline{\lambda}(Q_{\mathrm{m}}) |\hat{e}(t)| - 2||P_{\mathrm{m}} L C_{\mathrm{s}}|| \, |\tilde{x}(t)| \Big] \end{aligned}$$

so that $\dot{\mathcal{V}}(\hat{e}(t)) < 0$ for all $|\hat{e}(t)| > rv$, since $\tilde{x}(t)$ is UUB by r and v is given by (55). Hence, it follows from Lemma 4.1 that $|e(t)|$ is UUB by $r(1+v)$. □

Remark 4.4. It follows from Corollaries 4.1 and 4.2 that $|e(t)|$ is UUB by $r_{e_1} \equiv r(1+v)$, where:

$$r_{e_1} = \sqrt{\frac{\overline{\lambda}(P)\Psi_1^2 + \rho\tau\Psi_2^2}{\underline{\lambda}(\tilde{P})}} \left[1 + \frac{2||P_{\mathrm{m}} L C_{\mathrm{s}}||}{\underline{\lambda}(Q_{\mathrm{m}})} \right] \tag{58}$$

V. Illustrative Examples

This section gives several examples of the architecture described in Figure 1 using a model of wing rock dynamics[36]. Wing rock is a nonlinear phenomenon in which an aircraft exhibits an oscillation in roll at high angles of attack. A two state model for wing rock dynamics can be written in the form given by

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \big[u(t) + \Delta(x(t)) \big] \tag{59}$$

$$y_{\mathrm{s}}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + w(t) \tag{60}$$

where

$$\Delta(t, x(t)) = \big[\alpha_1 + f_1(t)\big] x_1(t) + \big[\alpha_2 + f_2(t)\big] x_2(t) + \alpha_3 |x_1(t)| x_2(t) + \alpha_4 |x_2(t)| x_2(t) + \alpha_5 x_1^3(t) + d(t) \tag{61}$$

with *constant* coefficients $\alpha_1 = 0.2314$, $\alpha_2 = 0.6918$, $\alpha_3 = -0.6245$, $\alpha_4 = 0.0095$, and $\alpha_5 = 0.0214$, and *time varying* coefficients $f_1(t)$ and $f_2(t)$, and an external disturbance $d(t)$. In (59) and (60), $x_1(t)$ represents the roll angle and $x_2(t)$ represents the roll rate, and $w(t)$ is measurement noise disturbance. The reference model is selected to be second order with a natural frequency of 1.6 rad/sec and a damping ratio of 0.8, and to have a unity gain from $r(t)$ to $y_{\mathrm{m}}(t)$ at low frequency. This amounts to choosing $K_1 = [2.56, 2.56]$ and $K_2 = 2.56$. We also chose $L^{\mathrm{T}} = [12.8 \ 64.0]$, which corresponds to $k = 5$ in (26). For the adaptive control design, we used sigmoidal basis functions of the form $\beta(x(t)) = \big[0.5, \ \beta_1(x_1(t)), \ \beta_2(x_2(t))\big]$, where $\beta_i(x_i(t)) = \frac{1-e^{-x_i(t)}}{1+e^{-x_i(t)}}$, $i = 1, 2$. Since $|\beta_i(x)| \leq 1$, it follows that $\bar{\beta} = 1.5$. For $\mu = 0.05$ and $Q_0 = 0.25 I_2$, it was determined by Lemma 3.2 that $\bar{\nu} = 124.6$ which implies that the adaptation gain in (15) must satisfy $\kappa_2 < 35.4$. Furthermore, we set $\Omega_1 = 0.95 I_3$, $\kappa_2 = 35$, and $\tau = 0.01$ seconds. We consider the command tracking problem in the following subsections with the initial conditions for the dynamics in (59) set zero.

Figure 3 compares the nominal (adaptation off) and adaptive control responses for the case when the ideal weights are constant $\big(f_1(t) = f_2(t) = 0\big)$, and in the absence of disturbances $\big(d(t) = w(t) = 0\big)$. Overall performance is significantly improved with adaptation. Improvement is measured by comparing how well the responses follow $y_{\mathrm{m}}(t)$. Figure 4 compares the responses for the case when the ideal weights are time varying, with $f_1(t)$ being a square wave having an amplitude of 0.5 and a period of 15 seconds and $f_2(t) = 0.5\sin(1.5t)$. The improvement with adaptation is greater in comparison to the improvement in Figure 3. Note that the effect of $f_1(t)$ is clearly evident in the nominal case.

Next we include the effect of a process disturbance and measurement noise. In this case $d(t)$ is a square wave having an amplitude of 0.1 and a period of 6 seconds. To model sensor noise we let $w(t)$ be a band-limited white noise process with a correlation time constant of 0.01 seconds, and a noise power level of 0.0001. These processes are depicted in Figure 5. Figure 6 shows that tracking performance is significantly improved with adaptation, and that the control time history is well behaved (sensor noise is not amplified). Figure 7 includes a comparison between $\Delta(t, x(t))$ and $u_{\mathrm{ad}}(t)$.

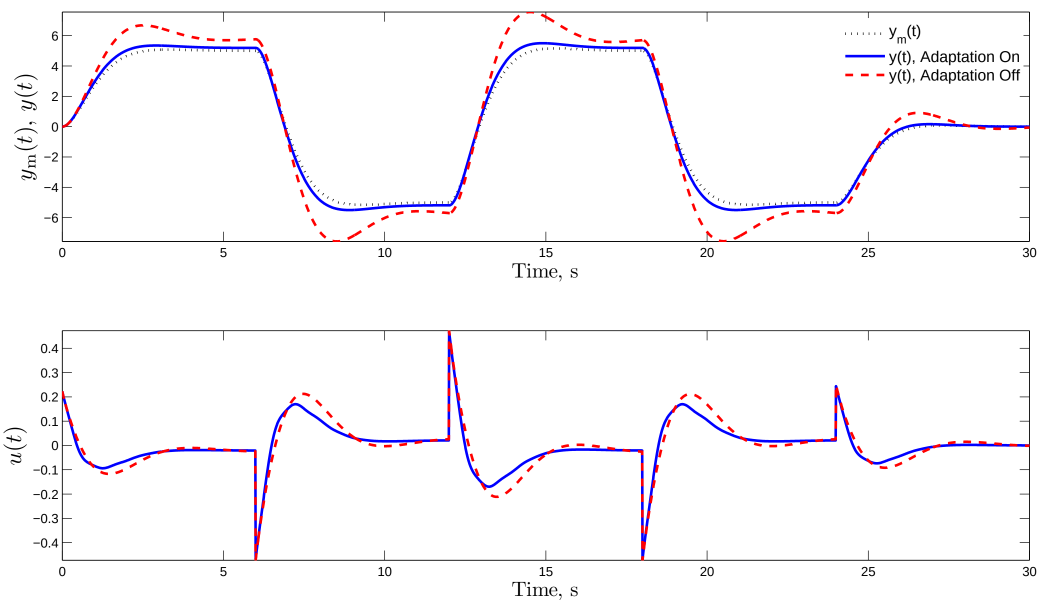

Figure 3. Nominal and adaptive control responses for the case of constant ideal weights

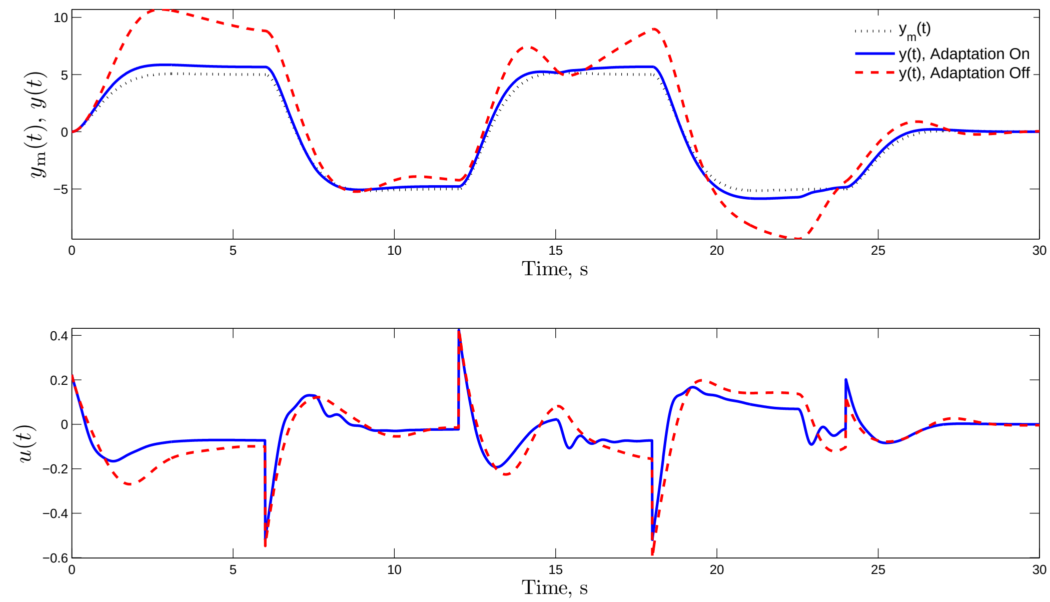

Figure 4. Nominal and adaptive control responses for the case of time varying ideal weights

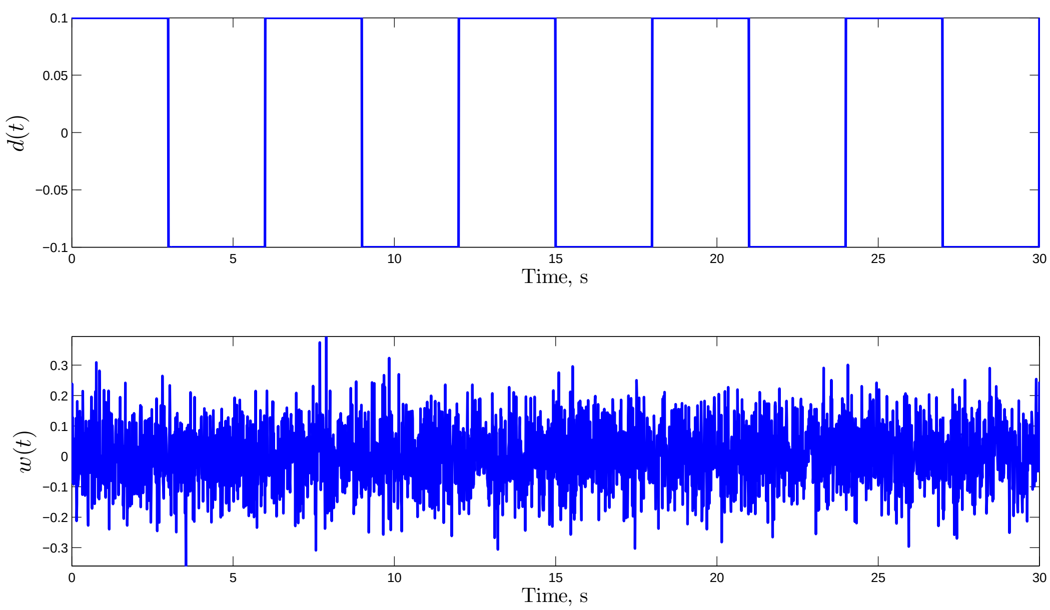

Figure 5. Depiction of $d(t)$ and $w(t)$

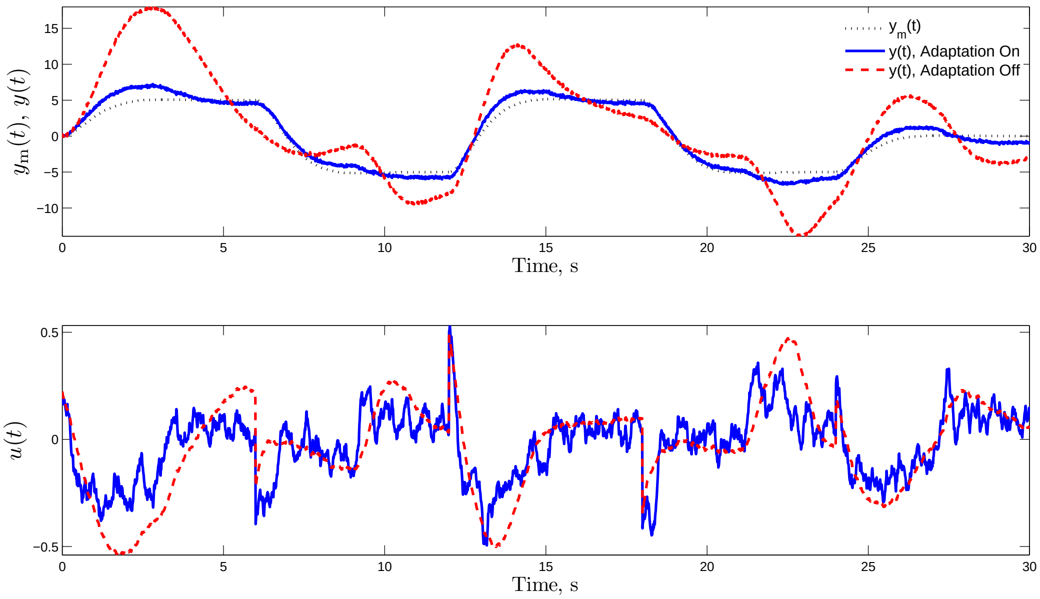

Figure 6. Nominal and adaptive control responses with disturbances for the case of time varying ideal weights

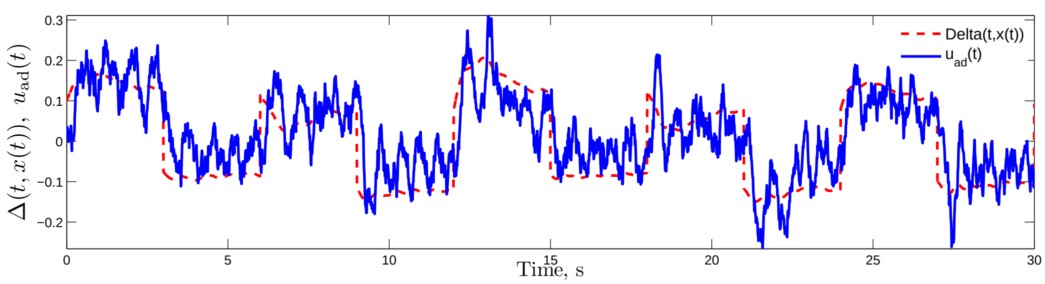

Figure 7. $\Delta(t, x(t))$ **and** $u_{\mathrm{ad}}(t)$

VI. Conclusions

This paper extends a previously developed state feedback, derivative-free adaptive control law to an output feedback form. The controller is particularly useful for situations in which the nature of the system uncertainty cannot be adequately represented by a set of basis functions with constant ideal weights, or for situations in which the ideal weights change abruptly, or there is difficult to model disturbance process. The level of complexity of the output feedback form is far less than many other methods, and it can be implemented in a form that augments an observer based linear controller architecture. Illustrative examples on a model of wing rock dynamics show that the presented theory and the simulation results are compatible.

Appendix A: Derivation of (43)

This appendix shows the derivation of (43) from (41). For this purpose, we identically write (41) as:

$$\begin{aligned}
\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) = & -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) + 2\tilde{x}^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(\hat{x}(t)) \\
& \boxed{-2\tilde{x}^{\mathrm{T}}(t)PB\hat{\Omega}_2^{\mathrm{T}}(t)\beta(\hat{x}(t))}^{\,\mathrm{a}} + 2\tilde{x}^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}\beta(\hat{x}(t)) \\
& +\rho\ \mathrm{tr}\big[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t) - \tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau) + \eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau) \\
& \boxed{+\eta\hat{\Omega}_2^{\mathrm{T}}(t)\hat{\Omega}_2(t)}^{\,\mathrm{a}} + \eta\Omega_2^{\mathrm{T}}\Omega_2 \ \boxed{-2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_1\tilde{W}(t-\tau)}^{\,\mathrm{a}} \\
& \boxed{+2\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_2}^{\,\mathrm{b}} \ \boxed{-2\eta\hat{\Omega}_2^{\mathrm{T}}(t)\Omega_2}^{\,\mathrm{a}}\big]
\end{aligned}$$

using (38) for the terms in $\boxed{\cdot}^{\,\mathrm{a}}$ and (42) for the term in $\boxed{\cdot}^{\,\mathrm{b}}$ produces:

$$\begin{aligned}
\dot{\mathcal{V}}(\tilde{x}(t),\tilde{W}_t) \leq & -\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + 2\tilde{x}^{\mathrm{T}}(t)PBg(x(t),\hat{x}(t)) \ \boxed{+2\tilde{x}^{\mathrm{T}}(t)PB[\Omega_1\tilde{W}(t-\tau)]^{\mathrm{T}}\beta(\hat{x}(t))}^{\,\mathrm{c}} \\
& \boxed{-2\tilde{x}^{\mathrm{T}}(t)PB[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)PB]^{\mathrm{T}}\beta(\hat{x}(t))}^{\,\mathrm{f}} \ \boxed{-2\tilde{x}^{\mathrm{T}}(t)PB[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)N]^{\mathrm{T}}\beta(\hat{x}(t))}^{\,\mathrm{d}} \\
& \boxed{+2\tilde{x}^{\mathrm{T}}(t)PB\Omega_2^{\mathrm{T}}\beta(\hat{x}(t))}^{\,\mathrm{e}} + \rho\ \mathrm{tr}\left[-\xi\tilde{W}^{\mathrm{T}}(t)\tilde{W}(t)\right] \ \boxed{+\rho\ \mathrm{tr}\left[-\tilde{W}^{\mathrm{T}}(t-\tau)\tilde{W}(t-\tau)\right]}^{\,\mathrm{g}} \\
& \boxed{+\rho\ \mathrm{tr}\left[\eta\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\right]}^{\,\mathrm{g}} \ \boxed{+\rho\ \mathrm{tr}\left[\eta\kappa_2^2B^{\mathrm{T}}P\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\right]}^{\,\mathrm{f}} \\
& \boxed{+2\rho\ \mathrm{tr}\left[\eta\kappa_2^2B^{\mathrm{T}}P\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)N\right]}^{\,\bar{\mathrm{d}}} + \rho\ \mathrm{tr}\left[\eta\kappa_2^2N^{\mathrm{T}}\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x})\beta(\hat{x})\tilde{x}^{\mathrm{T}}(t)N\right] \\
& \boxed{+\rho\ \mathrm{tr}\left[\eta\Omega_2^{\mathrm{T}}\Omega_2\right]}^{\,\mathrm{h}} \ \boxed{+\rho\ \mathrm{tr}\left[-2\eta\left[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\right]^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\right]}^{\,\bar{\mathrm{c}}} \\
& +\rho\ \mathrm{tr}\left[-2\eta\left[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)N\right]^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\right] \ \boxed{+\rho\ \mathrm{tr}\left[\gamma\tilde{W}^{\mathrm{T}}(t-\tau)\Omega_1^{\mathrm{T}}\Omega_1\tilde{W}(t-\tau)\right]}^{\,\mathrm{g}} \\
& \boxed{+\rho\ \mathrm{tr}\left[\eta^2\Omega_2^{\mathrm{T}}\Omega_2/\gamma\right]}^{\,\mathrm{h}} \ \boxed{+\rho\ \mathrm{tr}\left[-2\eta\left[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)PB\right]^{\mathrm{T}}\Omega_2\right]}^{\,\bar{\mathrm{e}}} \\
& +\rho\ \mathrm{tr}\left[-2\eta\left[\kappa_2\beta(\hat{x}(t))\tilde{x}^{\mathrm{T}}(t)N\right]^{\mathrm{T}}\Omega_2\right]
\end{aligned}$$

Since $\kappa_2 = 1/\rho\eta$, $\boxed{\cdot}^{\,\mathrm{c}}$ cancels $\boxed{\cdot}^{\,\bar{\mathrm{c}}}$, $\boxed{\cdot}^{\,\mathrm{d}}$ cancels $\boxed{\cdot}^{\,\bar{\mathrm{d}}}$, and $\boxed{\cdot}^{\,\mathrm{e}}$ cancels $\boxed{\cdot}^{\,\bar{\mathrm{e}}}$ in the above expression. Furthermore, grouping the terms in $\boxed{\cdot}^{\,\mathrm{f}}$ as $-\kappa_2\tilde{x}^{\mathrm{T}}(t)PBB^{\mathrm{T}}P\tilde{x}(t)\beta^{\mathrm{T}}(\hat{x}(t))\beta(\hat{x}(t))$, the terms in $\boxed{\cdot}^{\,\mathrm{g}}$ as $-\rho\ \mathrm{tr}\big[\tilde{W}^{\mathrm{T}}(t-\tau)[I-(\eta+\gamma)\Omega_1^{\mathrm{T}}\Omega_1]\tilde{W}(t-\tau)\big]$, and the terms in $\boxed{\cdot}^{\,\mathrm{h}}$ as $\rho(\eta+\eta^2/\gamma)\ \mathrm{tr}[\Omega_2^{\mathrm{T}}\Omega_2]$ yields (43).

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