An LMI-based Analysis of Stability Margins in Adaptive Flight Control

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In this paper we show that, by casting the error dynamics of an adaptive control system into a Linear Parameter Varying form, we can apply linear matrix inequality techniques and calculate stability margins for adaptive control systems. More precisely, by taking the viewpoint that nonlinear systems, in particular some classes of adaptive controllers, can be parameterized by a set of linear systems, we show that stability margins defined in linear robust control theory can be analyzed. Whereas the concept of gain-margin is clear whether a system to be analyzed is linear or nonlinear, frequency responses for general nonlinear systems are not defined, and therefore in this paper we consider time-delay margin instead of phase-margin in our formulation. Theoretically guaranteed margins are verified using the Generic Transportation Model simulation provided by NASA Langley.

I. Introduction

This paper concerns the application of a Linear Matrix Inequality (LMI)-based tool that quantifies guaranteed stability margins for adaptive flight control systems. Despite numerous successes in applications and flight demonstrations,1, 2 adaptive control has not been accepted with sufficient confidence in manned systems. The reason for this is the lack of analysis tools that provide a guarantee for stability, a computable bound for stability margins, and a measure of performance degradation when basic assumptions in adaptive control are violated.3

Standard adaptive control methods4–9 employ time-varying parameters that do not fit the traditional validation process of stability and robustness for closed-loop flight control systems, which is mandatory for flight certification. The incorporation of time-varying adaptation laws fundamentally changes the characterization of stability for adaptive systems (even when the process under control is linear) from exponential stability to weaker assurances such of either asymptotic stability or uniform ultimate boundedness (UUBness) of the tracking errors.6 The asymptotic nature of stability analysis has been a fundamental obstacle in ensuring robustness of adaptive control. Whereas exponential stability is a prerequisite for Lyapunov’s first theorem, in adaptive control, this can

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only be attained under highly restrictive persistency of excitation condition. Consequently, it can
not be claimed that adaptive control is robust to unmatched uncertainties, unmodeled dynamics,
external disturbances.¹⁰

Employing modification terms in adaptive laws, such as \( \sigma \)-modification,¹¹ \( e \)-modification,⁴ and
projection,¹² weakens the notion of stability from asymptotic convergence to UUBNess of tracking
errors.¹³ This notion is useful for neural network (NN)-based adaptive algorithms¹³⁻¹⁵ because the
inherent network approximation error necessitates the employment of modifications for proof of
boundedness for closed-loop signals. In the case of a NN-based adaptive algorithm, this has been
the price that one has to pay in order to eliminate the requirement of having a perfectly known
regression vector. Nevertheless, even though incorporating modification terms in adaptive laws has
shown to be robust to a limited class of modeling errors that violates the standard assumptions
in adaptive control,⁶ a general framework to quantitatively analyze the performance of adaptive
control systems with respect to unstructured uncertainties and unmodeled dynamics has not yet
been developed.

In this paper, we consider analysis of an adaptive control system that employs \( \sigma \)-modification.
Equipped with the notion of UUBNess, we show that the combined error dynamics, composed of
the tracking error and the weight estimate error, can be cast as an exponentially stable system
under bounded perturbation, which has been a standard result in Lyapounov stability analysis.
Our viewpoint, however, deviates from the standard Lyapunov analysis in that the exponentially
perturbed system can also be viewed as a linear-time varying (LTV) system under bounded per-
turbation. Moreover, we show that by restricting some variables into a known compact domain,
the LTV system can be parameterized as a linear-parameter varying (LPV) system for which well
developed LMI analysis tools exist. In this setting, the overall formulation resembles nonlinear sys-
tem analysis via linear differential inclusions.¹⁶ In other words, we do not attempt to linearize the
adaptive system in its steady-state as was done in Ref.s [17, 18]. The resulting framework therefore
enables us to compute theoretically guaranteed rate of convergence, accuracy of the tracking error
both in transient and steady state, which are guaranteed to be less conservative than the standard
analysis in the literature,¹⁹ and stability margins in terms of definitions given in the robust control
theory, all within an adaptive control setting.

The paper is organized as follows. In Section II, we present a standard \( \sigma \)-modification based
adaptive control method. In Section III, we show how the combined error dynamics can be cast
into a LPV form. In Section IV, we present LMI tools that can analyze the nominal stability
characteristics as well as stability margins for the adaptive controller with \( \sigma \)-modification. The
approach is illustrated by applying the LMI tools to the problem of roll rate regulation using NASA’s
Generic Transportation Model (GTM) in Section V. Conclusions and future research directions are
given in Section VI.

II. Adaptive Control with \( \sigma \)-modification

Consider a single-input single-output system described by:

\[
\dot{x}(t) = Ax(t) + b(u(t) + W^\top \phi(x(t))) \\
y(t) = c^\top x(t),
\]

where \( x(t) \in \mathbb{R}^n \) is the system state vector, \( u(t) \in \mathbb{R} \) is the input, \( y(t) \in \mathbb{R} \) is the output, \( W \in \mathbb{R}^N \)
is a uncertain parameter vector, \( \phi(x(t)) \in \mathbb{R}^N \) is a known set of smooth basis functions, and
the system matrices \( A, b, c^\top \) are known. A nominal linear controller:

\[
u_{nom}(t) = -K_x^\top x(t) + K_r r(t),
\]
is assumed to be designed such that the resulting closed-loop system with the known part of the system in (1) satisfies design specifications. Hence we can define a reference model for the desired behavior using

\[ \dot{x}_m(t) = A_m x_m(t) + b_m r(t) \]
\[ y_m(t) = c^T x_m(t), \]

where \( A_m = A - bK_x^T \) is Hurwitz, \( b_m = bK_r \), and \( r(t) \) is a bounded reference command.

Let

\[ u(t) = u_{nom}(t) - u_{ad}(t), \]

where \( u_{ad}(t) \) is an adaptive signal introduced to approximately cancel the uncertainty \( W^T \phi(x(t)) \):

\[ u_{ad}(t) = \hat{W}(t)^T \phi(x(t)), \]

whose estimate \( \hat{W}(t) \) for the ideal weight \( W \) in (1) is updated using:

\[ \dot{\hat{W}}(t) = -\gamma \phi(x(t)) e(t)^T P b, \]

where \( \gamma > 0 \in \mathbb{R} \) is the adaptation gain, and \( P > 0 \) is obtained by solving the following Lyapunov function with a selected \( Q > 0 \):

\[ A_m^T P + P A_m + Q = 0. \]

Stability results in adaptive control are cast in terms of the tracking error:

\[ e(t) = x_m(t) - x(t), \]

whose dynamics are described by:

\[ \dot{e}(t) = A_m e(t) + b\hat{W}(t)^T \phi(x(t)), \]

where \( \hat{W}(t) = W(t) - W \) is the weight estimation error.

Let

\[ \zeta(t) = [e(t)^T, \hat{W}(t)^T]^T. \]

Then the error dynamics composed of the tracking error and the weight estimation error are described by:

\[ \dot{\zeta}(t) = \begin{bmatrix} A_m & b\phi(x(t))^T \\ -\gamma \phi(x(t)) b^T P & 0 \end{bmatrix} \zeta(t), \]

where \( \gamma > 0 \in \mathbb{R} \) is the adaptation gain, and \( P > 0 \) is obtained by solving the following Lyapunov function with a selected \( Q > 0 \):

\[ A_m^T P + P A_m + Q = 0. \]

A stability analysis for the system in (11) is typically carried out by considering the following Lyapunov candidate function:

\[ V(\zeta(t)) = e(t)^T Pe(t) + \frac{1}{\gamma} \hat{W}(t)^T \hat{W}(t) = \zeta(t)^T \begin{bmatrix} P & 0 \\ 0 & \gamma^{-1} I_N \end{bmatrix} \zeta(t), \]

whose time derivative of \( V \) along with (8) and (6) reduces to:

\[ \dot{V}(\zeta(t)) = -e(t)^T Q e(t) \leq 0. \]

This guarantees that \( \zeta(\cdot) \in L_\infty \). Further analysis shows that 1) \( e(\cdot) \in L_2 \cap L_\infty \), and the tracking error globally converges to the zero via Barbalat’s lemma \( (e(t) \to 0 \text{ as } t \to \infty) \) 2) the weight errors remain bounded \( \hat{W}(\cdot) \in L_\infty \), and \( \lim_{t \to \infty} \hat{W}(t) = \lim_{t \to \infty} \hat{W}(t) = 0 \). This result implies
that the only guaranteed stability results for adaptive control are asymptotic convergence of the
tracking error to zero and the boundedness of the weight estimate, which is weaker than exponential
stability in linear system theory. As a result, unlike exponentially stable systems which are robust
to a small perturbation, adaptive control system cannot be guaranteed to remain bounded when
perturbed. A prominent result in this regard concerns the so-called parameter drift phenomenon in
the presence of a bounded external disturbance. Whereas stable linear systems remain bounded in
the presence of bounded disturbances, an adaptive control system may exhibit a drift in
infinity . Therefore modification terms are required to prevent parameter drift in adaptive systems,
and various modifications have been proposed.\(^6\)

In our effort, we employ a modification term as an essential ingredient of adaptive control
and study the stability properties of adaptive control systems. More specifically, we introduce a
\(\sigma\)–modification, so that the update law (6) is modified to

\[
\hat{W}(t) = -\gamma \phi(x(t))e(t)^\top Pb + \sigma \hat{W}(t),
\]

which is equivalently written in terms of \(\hat{W}(t)\) as

\[
\hat{W}(t) = -\gamma \phi(x(t))e(t)^\top Pb - \sigma \hat{W}(t) - \sigma W,
\]

With this modification, stability analysis using the Lyapunov candidate in (12) leads to:

\[
\dot{V}(\zeta(t)) = -e(t)^\top Qe(t) - 2\sigma \hat{W}(t)^\top \hat{W}(t)
\]
\[
= -e(t)^\top Qe(t) - \frac{\sigma}{\gamma} \left( \|\hat{W}(t)\|^2 + \|\hat{W}(t)\|^2 - \|W\|^2 \right)
\]
\[
\leq -\lambda_{\text{min}}(Q) \|e(t)\|^2 - \frac{\sigma}{\gamma} \|\hat{W}(t)\|^2 + \frac{\sigma}{\gamma} \|W\|^2
\]
\[
\leq -\mu \|\zeta(t)\|^2 + \frac{\sigma}{\gamma} \|W\|^2,
\]

where \(\mu = \min\{\lambda_{\text{min}}(Q), \sigma\}\). Whenever \(\|\zeta(t)\| \geq \sqrt{\frac{\sigma}{\gamma \mu}} \|W\|\), \(\dot{V} \leq 0\). Hence, \(\zeta(t)\) is UUB.\(^{20}\) Note
that by employing \(\sigma\)–modification, the resulting stability proof has weakened to UUBness of the
closed-loop signals, however, the adaptive system remains bounded in the presence of bounded
external disturbances. Due to the modification, the combined error \(\zeta(t)\) evolves according to:

\[
\dot{\zeta}(t) = \begin{bmatrix}
A_m \\
-\gamma \phi(x(t))b^\top P \\
A(x(t))
\end{bmatrix}
\begin{bmatrix}
b \phi(x(t)) \end{bmatrix}
\begin{bmatrix}
0 \\
-\sigma I_N \\
B
\end{bmatrix}
W
\]

\[
e(t) = \begin{bmatrix}
I_n \\
0_{n \times N}
\end{bmatrix}
\tilde{C}_e
\begin{bmatrix}
\zeta(t).
\end{bmatrix}
\]

III. Affine Parametrization with an Uncertainty in the Input Channel

The concept of stability margin for adaptive control is formulated by considering an uncertain
mapping \(\Delta_p(\cdot) : \mathcal{L}_1^{\infty} \rightarrow \mathcal{L}_2^{\infty}\) in the feedback configuration shown in Figure 1, in which \(\mathcal{L}_2^{\infty}\) is a
one dimensional \(\mathcal{L}_2^{\infty}\) extended space.\(^{20}\) It will be shown that the feedback interconnection depicted
in Figure 1 allows us to derive the same stability margins considered in the non-adaptive robust
control literature.\(^{21}\)
Figure 1. Adaptive control with uncertainty in the input channel

A. An LTV form for adaptive control

With the objective of formulating a LPV system for the control system in Figure 1, we note that

\[
\begin{align*}
z_p(t) &= K_r r(t) - K_x x(t) - \hat{W}(t)^\top \phi(x(t)) + w_p(t) \\
&= K_r r(t) - K_x x_m(t) + e(t) - \hat{W}(t)^\top \phi(x(t)) + w_p(t) \\
&\quad - W^\top \phi(x(t)).
\end{align*}
\]

Let \( f(x(t)) = W^\top \phi(x(t)) \). Then, by the mean value theorem, we have:

\[
\begin{align*}
f(x(t)) &= f(x_m(t)) + \frac{\partial f}{\partial x} |_{x(t)} e(t) \\
&= W^\top \phi(x_m(t)) + W^\top \phi_x(x(t)) e(t).
\end{align*}
\]
where $\bar{x}(t) = x_m(t) + \theta e(t), \ \theta \in [0, 1]$. Therefore, in the presence of uncertainty in the input channel, we have the following dynamics:

$$
\dot{\zeta}(t) = \begin{bmatrix}
A_m & b^\top \phi(x(t)) \\
-\gamma \phi(x(t)) b^\top P & -\sigma I_N
\end{bmatrix} \zeta(t) + \begin{bmatrix}
0 \\
-\sigma I_N
\end{bmatrix} W + \begin{bmatrix}
b \\
0
\end{bmatrix} w_p(t)
$$

$$
z_1(t) = \begin{bmatrix}
-K_x - W^\top \phi_x(\bar{x}(t)) & -\phi(x(t))^\top
\end{bmatrix} \zeta(t) + \bar{D} w_p(t),
$$

$$
z_p(t) = z_1(t) + \left( K_r r(t) - K_x \bar{x}_m(t) - W^\top \phi(\bar{x}_m(t)) \right),$$

$$
w_p(t) = \Delta(z_p(t)),$$

$$
e(t) = \begin{bmatrix}
I_n & 0_{n \times N}
\end{bmatrix} \zeta(t).$$

where $\phi_x$ denotes the partial derivative respect to $x$, and $\bar{D} = 1$. Let $w_o = W$. We note that $w_0, z_0 \in L_\infty \cap L_2$ because $r(t)$ and $x_m(t)$ are bounded for $\forall t \geq 0$. By introducing a system $\Sigma$ as the one having $\zeta$ as the state vector, $w_o$ and $w_p$ as inputs, and $z_1$ as output in (20), the overall closed loop system in (20) can be described as the feedback interconnection depicted in Figure 2.

![Figure 2. Interconnection with the pull-out of the uncertainty](image)

**B. Affine parametrization**

From (17), we can see that in the absence of the uncertain mapping $\Delta(\cdot)$, affine parametrization is feasible by viewing $\rho(t) = \phi(x(t))$ on a compact domain $\Omega_x$ in which $x(t)$ is assumed to belong. For analysis of nominal stability characteristics, such as convergence rate, domain of attraction, the size of an uniform ultimate bound and the transient behavior, this parametrization is sufficient.
However, to incorporate the mapping $\Delta(\cdot)$ in the input channel, we need to take into account the fact that the system in (20) involves additional uncertainty $W^T \phi(x(t))$. We treat this as additional parameter in our analysis and hence define the following augmented parameter vector:

$$\rho^a(t) = \begin{bmatrix} \rho(x(t)) \\ W^T \phi_x(\bar{x}(t)) \end{bmatrix} \in \mathbb{R}^{N+n}$$

and a set to which the parameter belongs:

$$\mathcal{P}^a := \{ \rho^a = (\rho_1^a, \ldots, \rho_{N+n}^a) : \rho_i^a \in [\underline{\rho}_j^a, \overline{\rho}_j^a]\},$$

whose corners belong to the following set:

$$\mathcal{P}_0^a := \left\{ \rho^a = (\rho_1^a, \ldots, \rho_{N+n}^a) : \rho_i^a \in [\underline{\rho}_j^a, \overline{\rho}_j^a]\right\}.$$  \hspace{1cm} (23)

As in Ref.[19], the corners of the first $N$ parameters are found by $\underline{\rho}_j^a = \min(\phi_j(x))$, $\overline{\rho}_j^a = \max(\phi_j(x))$, $1 \leq j \leq N$, on the known compact domain $\Omega_x$. The other parameter $W^T \phi_x(\bar{x}(t))$ is unknown because it involves the unknown parameter $W$ and the unknown variable $\bar{x}(t)$. It is assumed that the uncertainty satisfies a linear growth assumption with a known upper bound.

**Assumption 1.** \[ \| \frac{\partial}{\partial x} W^T \phi(x) \| \leq w^* \text{ for } \forall x \in \Omega_x. \]

Since $\bar{x}(t) = x_m(t) + \theta e(t)$ with $\theta \in [0,1]$ and $\bar{x}(t) \in \Omega_x$ as long as $x(t) \in \Omega_x$, Assumption 1 ensures that $\| W^T \phi_x(\bar{x}(t)) \| \leq w^*$. Then $|\rho_j^a| \leq \rho_j^a(N+1 : N+n) \leq \| \rho^a(N+1 : N+n) \| \leq w^*$. This leads to $\rho_j^a = -w^*, \overline{\rho}_j^a = w^*$ for $\forall N+1 \leq j \leq N+n$.

With the parameter vector $\rho^a(t)$, we have

$$\bar{A}(x(t)) = \bar{A}(\rho^a(t)) = A_0 + \sum_{j=1}^{N+n} \rho_j^a(t) A_j,$$

$$\bar{C}(x(t), x(t)) = \bar{C}(\rho^a(t)) = C_0 + \sum_{j=1}^{N+n} \rho_j^a(t) C_j,$$

where $A_0 = \begin{bmatrix} A_m & 0_{n \times N} \\ 0_{N \times N} & -\sigma I_N \end{bmatrix}$, $A_j \in \mathbb{R}^{(n+N) \times (n+N)}$ is a matrix such that $A_{j}(1:n,k) = b$, $A_{j}(k,1:n) = -\gamma b^T P$ if $k = j$, and $A_{j}(k,l) = 0$ otherwise ($k \neq j$ nor $l \neq j$), $C_0 = \begin{bmatrix} -K_x & 0_{1 \times N} \end{bmatrix}$, $C_j \in \mathbb{R}^{1 \times (N+n)}$ such that $C_{j}(n+k) = -1$ if $k = j$, and $C_{j}(k) = 0$ otherwise for $1 \leq j \leq N$, and $C_{j}(k) = -1$ if $k = N+j$, $C_{j}(k) = 0$ otherwise for $N+1 \leq j \leq N+n$. The notation $1:n$ is used to represent indices from 1 to $n$.

**IV. LMI Analysis**

**A. Nominal analysis**

In the absence of the uncertain mapping $\Delta(\cdot)$, the system in (20) reduces to the system in (17). For this case, the parameter in (21) is reduced to $\rho(x(t)) \in \mathbb{R}^N$ and belongs to $\mathcal{P} := co(\mathcal{P}_0)$, where $\mathcal{P}_0 := \{ \rho = (\rho_1, \ldots, \rho_N) : \rho_j \in [\underline{\rho}_j, \overline{\rho}_j]\}$. Lemmas from Ref.[19] can be employed to analyze stability, convergence rate, transient response, and the ultimate bound for the tracking error.
Lemma 1. Suppose that there exists \( X = X^T > 0 \), \( \mu > 0 \) such that
\[
\bar{A}(\rho)^T X + X \bar{A}(\rho) < -\mu I, \quad \forall \rho \in \mathcal{P}_0.
\] (25)
Then the system in (17) is UUB.

Lemma 2. Suppose that there exists \( X = X^T > 0 \), \( \mu > 0 \) such that
\[
\bar{A}(\rho)^T X + X \bar{A}(\rho) < -\mu X, \quad \forall \rho \in \mathcal{P}_0.
\] (26)
Then \( \zeta(t) \) in (17) is exponentially bounded by:
\[
\|\zeta(t)\| \leq \|\zeta(0)\| e^{-\frac{\mu}{2}t} + 2\sqrt{\kappa(X)}\sigma \|W\| \left[1 - e^{-\frac{\mu}{2}t}\right],
\] (27)
where \( \kappa(X) = \lambda_{\max}(X)/\lambda_{\min}(X) \).

The convergence rate \( \mu \) can also indicate the degree to which norm-bounded unmatched uncertainty can be tolerated because the LMI in (26) guarantees \( \bar{A}(t) + \frac{\mu}{2} I < 0 \).

Lemma 3. Suppose that there exists \( X = X^T > 0 \), \( \mu > 0 \), and \( \beta_1, \beta_2, \nu > 0 \) such that
\[
\begin{bmatrix}
\bar{A}(\rho)^T X + X \bar{A}(\rho) + \mu X & XB \\
B^T X & -\nu I_N
\end{bmatrix} < 0,
\]
\begin{bmatrix}
\mu X & 0 & \bar{C}^T \\
0 & (\beta_2 - \nu)I & 0 \\
\bar{C} & 0 & \beta_1 I_N
\end{bmatrix} > 0, \quad \forall \rho \in \mathcal{P}_0,
\] (28)
Then the tracking error is upper bounded by:
\[
\|e(t)\| \leq \sqrt{\beta_1 \mu \lambda_{\max}(X)} \|\zeta(0)\| e^{-\frac{\mu}{2}t} + \sqrt{\beta_1 \beta_2 \sigma} \|W\|.
\] (29)

B. Stability margin analysis

In the presence of the uncertain mapping \( \Delta(\cdot) \), we are concerned with exponential stability of the feedback interconnection depicted in Figure 2. Therefore, we consider the system \( \Sigma \) by restricting its input and output to \( w_p \) and \( z_1 \):
\[
\dot{\zeta}(t) = \bar{A}(\rho^a(t))\zeta(t) + \bar{B}_p w_p(t), \quad \rho^a(t) \in \mathcal{P}^a
\]
\[
z_1(t) = \bar{C}(\rho^a(t))\zeta(t) + \bar{D} w_p(t),
\] (30)
and obtain its \( L_2 \) gain by the following Lemma.

Lemma 4. The system in (30) has \( L_2 \) gain less than \( \gamma_1 \) if there exists \( X = X^T > 0 \) such that
\[
\begin{bmatrix}
\bar{A}(\rho^o)^T X + X \bar{A}(\rho^o) + \bar{C}(\rho^o)^T \bar{C}(\rho^o) & X \bar{B}_p + \bar{C}(\rho^o)^T \bar{D} \\
\bar{B}_p^T X + \bar{D}^T \bar{C}(\rho^o) & \bar{D}^T \bar{D} - \gamma_1^2 I
\end{bmatrix} < 0, \quad \forall \rho \in \mathcal{P}_0^o.
\] (31)

That is, for the system in (20), we have:
\[
\|z_{pr}\|_{L_2} \leq \gamma_1 \|w_{pr}\|_{L_2} + \beta_p, \quad \forall \tau \geq 0,
\] (32)
for \( \forall z_o \in \mathcal{L}_{2e} \) and \( \beta_p \geq 0 \) if \( w_0 = 0 \). The signal \( w_{pr} \) represents the truncated signal defined by \( w_{pr}(t) = w_p(t), \) \( 0 \leq t \leq \tau \) and \( w_{pr}(t) = 0, \) \( t > \tau \). Now, suppose that the uncertain mapping \( \Delta(\cdot) : \mathcal{L}_{2e} \rightarrow \mathcal{L}_{2e} \) satisfies
\[
\|w_{pr}\|_{L_2} \leq \gamma_2 \|z_{pr}\|_{L_2} + \beta_{\Delta}, \quad \forall \tau \geq 0,
\] (33)
where $\beta_{\Delta} \geq 0$. Then, the closed-loop system remains stable and the all the signals are bounded if $\gamma_1\gamma_2 < 1$, which follows from the small-gain theorem.\(^\text{22}\)

Equation (33) specifies the class of tolerable uncertain mappings in terms of an $L_2$ gain. That is, any static or dynamic mapping whose $L_2$ gain is less than $\frac{1}{\gamma_1}$ will not destabilize the adaptive control system in Figure 1. For consideration of standard margins, we note that Gain Margin (GM) and Phae Margin (PM) are defined, based on the Nyquist stability criterion, on the frequency domain. Whereas stability in the time domain can lead to an equivalent condition in the frequency domain for LTI systems, frequency responses for nonlinear systems are in general not defined. This implies that the concepts of GM/PM cannot be extended to nonlinear systems that can only be analyzed in the time domain. However, if we define GM as the allowable gain variation in the input channel (as is the case for the LTI system) and a delay margin (DM) as the maximal time-delay that does not result in instability when introduced in the input channel (as is the case for the LTI system), we can derive GM/DM by investigating the tolerable gain variation and time-delay that the class in (33) may induce in the input channel.

With GM/DM in mind, suppose that $\Delta(\cdot)$ is restricted to a LTI system. Then, we note that any LTI system whose $H_\infty$ norm is less than $\gamma_2$ satisfies (33). If $\Delta(\cdot)$ is a LTI mapping, Figure 3(a) leads to a transfer function $G_\Delta(s) = \frac{z_p(s)}{u(s)} = \frac{-1}{1+\Delta(s)}$. Figure 3(b) depicts a Nyquist plot for $-1 + \Delta(j\omega)$ as a circle centered at $-1 + j0$. Note that $\frac{1}{1+\gamma_2} \leq |G_\Delta(j\omega)| \leq \frac{1}{1-\gamma_2}$, and the induced phase shifts are $\pm \theta_m = \pm \sin^{-1} \sqrt{\frac{4-(\gamma_2^2-2)^2}{4}}$ when $|G_\gamma(j\omega)| = 1$. Therefore, if the input channel depicted in Figure 3(a) is connected to an underlying LTI system, and the closed-loop system remains stable, we obtain the following guaranteed margins:\(^{23}\)

\[
\begin{align*}
\text{GM}_{\text{increase}} &= 20 \log_{10} \left( \frac{1}{1 - \gamma_2} \right) (dB), \\
\text{GM}_{\text{decrease}} &= 20 \log_{10} \left( \frac{1}{1 + \gamma_2} \right) (dB), \\
\text{PM} &= \pm 57.3 \sin^{-1} \sqrt{\frac{4 - (\gamma_2^2 - 2)^2}{4}} (\text{degree})
\end{align*}
\]

Figure 4 depicts the block diagram in Figure 1 by incorporating the transfer function $G_\Delta(s)$ depicted in Figure 3(a). The viewpoint implied in Figure 4 is that in considering stability margins,
we break the loop at the input channel as is standard in the literature, in contrast to that we have obtained a class of tolerable uncertain mappings by breaking the loop at the location of \( w_p \) to apply the bounded real lemma in (31) in Figure 1. More specifically, we attempt to derive guaranteed GM/DM for the adaptive controller by utilizing the guaranteed margins in (34). The only subtlety in this pursuit is that the guarantees in (34) are only valid for a LTI system, but, the underlying system depicted in Figure 4 is time-varying.

For more rigorous treatment of this issue, as has been done in (20), we recast the dynamics in Figure 4 in terms of \( \{e(t), \bar{W}(t)\} \) as follows:

\[
\dot{\zeta}(t) = \begin{bmatrix} A - bW^T \phi_x(\bar{x}(t)) & 0 \\ -\gamma \phi(\bar{x}(t)) & -\sigma I_N \end{bmatrix} \begin{bmatrix} 0 \\ -\sigma I_N \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} w_o + \begin{bmatrix} -b \\ 0 \end{bmatrix} (z_p(t) - z_0(t)) \\

u(t) = \begin{bmatrix} -K_x - W^T \phi_x(\bar{x}(t)) \\ -\phi(\bar{x}(t))^T \end{bmatrix} \begin{bmatrix} A_L(\bar{x}(t)) \\ C(\bar{x}(t), \bar{x}(t)) \end{bmatrix} \zeta(t) + z_0(t), \\

z_p(t) = G_\Delta(s) u(t).
\]

Since the system in (35) is the same system as in (20), it is exponentially stable when \( w_0 = 0, z_0 = 0 \). Moreover, the interconnection is exponentially stable for any \( \rho^a \in \mathcal{P}^a \). That is, for any time instant \( t_0 \in [0, \infty) \), \( \bar{A}_L(t_0), \bar{B}_p, \bar{C}(t_0) \) and \( G_\Delta(s) \) are interconnected to be exponentially stable as long as \( \rho^a(t_0) \in \mathcal{P}^a \). Therefore, the margins in (34) are valid for every LTI system that is a snapshot loop-gain system for every time instant. A guaranteed DM can then be derived as the worst-case
time delay among LTI loop gains on the boundary of \( \text{co}(P_a) \). In other words, if we obtain the highest cross-over frequency \( \omega_{\text{cmax}} \) among boundary systems, the guaranteed DM is obtained by:

\[
DM = \frac{\pi}{180} \frac{|PM|}{\omega_{\text{cmax}} (\text{sec})}.
\] (36)

V. Simulations with the GTM model

We apply the analysis procedure for an adaptive controller designed for roll rate regulation using the GTM model. Consider the following SISO system:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + b(u(t) + W^T \phi(x(t))) + b_m p_{\text{cmd}}(t), \\
y(t) &= c^T x(t)
\end{align*}
\] (37)

where the state vector \( x = [p, v, p, r] \) is composed of an integrator of the roll rate error, the \( Y \)-axis velocity, the roll rate, and the yaw rate, respectively. The output \( y(t) \) represents the roll rate, the control signal \( u = \delta_a \) represents the aileron deflection, and \( p_{\text{cmd}}(t) \) represents the roll rate reference command. The system matrices in (37) are obtained by linearizing the GTM model at a selected trim point and are given by:

\[
A = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & -0.8532 & 6.5778 & -186.3175 \\
0 & -0.8720 & -8.7068 & 1.9306 \\
0 & 0.3365 & -0.2895 & -2.0953
\end{bmatrix},
\quad
b = \begin{bmatrix}
0 \\
-0.0665 \\
-1.7828 \\
-0.0462
\end{bmatrix}^T,
\quad
c = \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}^T,
\quad
b_m = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}^T.
\] (38)

The nominal controller in (2) is designed as a linear quadratic regulator (LQR) whose feedback and feedforward gains are:

\[
K_x^T = \begin{bmatrix}
100.0000 & 0.3868 & -6.5963 & -5.3349
\end{bmatrix},
\quad
K_r = 0.
\] (39)

The reference model in (3) is realized as a nominal closed loop system in which the known part of the linear system in (37) is regulated by the LQR controller. The closed-loop system has gain margin (GM) of infinity, the phase margin (PM) of 75.9873 ° at the crossover frequency of 14.7172 rad/s. This implies a time-delay margin (DM) of 0.09 second. Figure 5 shows the time responses of roll rate when the linear controller is applied to the nonlinear GTM simulation. While the nominal response shows good tracking performance, introducing a time delay of 0.09 second leads to an instability. The actual GTM model delay margin was found to be 0.075 second by trial and error, and the corresponding time response is shown in Figure 5(b).

The adaptive controller is realized by employing 4 neurons with sigmoidal functions of the form

\[
\phi_i(x) = \frac{1}{1 + e^{-x_i}}, \quad i = 1, \ldots, 4.
\] (40)

The matrix \( P \) in (6) is obtained by solving the Lyapunov equation in (7) with \( Q = \text{diag}\{0, 0, 5, 1\} \). Since the first state is due to employment of an integral control and hence does not induce any uncertainty, the matched parameterized uncertainty is assumed to satisfy \( \|W^T \phi(z)\| \leq 1 \) for \( \forall z = (v, p, r) \). Therefore, we perform the stability margin analysis in Section B with the following affine parameters:

\[
P^o := \{ \rho^o = (\phi(x)^T, W^T \phi(x)^T) : \rho_j^o \in [\underline{\rho}_j^o, \overline{\rho}_j^o] \},
\] (41)

where \( \underline{\rho}_j^o = 0, \overline{\rho}_j^o = 1 \) for \( j = 1, \ldots, 4 \) because \( 0 \leq \phi_i(x) \leq 1 \), and \( \rho_j^o = -1, \overline{\rho}_j^o = 1 \) for \( j = 5, \ldots, 7 \).

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The parametrization in (41) implies that the stability analysis is performed globally under the assumption that the description in (37) is valid for the entire domain $\mathbb{R}^4$. However, as the system in (37) is a linearized model with a matched parametric uncertainty around the trim point, we should not expect the stability margin result to be valid for the entire nonlinear GTM simulation model. Whereas the system in (37) never goes unstable with the given class of adaptive controller regardless of the adaptation gain, employing high-adaptation gains does make the nonlinear simulation go unstable. This indicates that the following points need to be taken into account for proper assessment of the analysis result. First, the instability that occurs with fast adaptation indicates the presence of uncertainties that violate the system description in (37), which can include neglected servo and engine dynamics. As a result, the validity of the analysis result depends on how closely the system model in (37) approximates the nonlinear GTM model. Second, while the full nonlinear simulation involves additional control loops in the pitch and yaw channels, we assume that the coupling effects due to these channels are negligible to simplify the analysis. In other words, we implicitly assume that the uncertainties that violate the problem formulation in (37), except gain variation and time-delay in the aileron channel, are negligible when we perform the stability margin analysis. A quantitative analysis for gain variation and allowable time delay in the presence of unmatched uncertainty requires further extension of the current formulation given in (37) and will be addressed in a future publication.

From the above considerations, we restrict our analysis to the cases of adaptation gains and $\sigma$—modification gains that do not lead to instability in the nominal case. Nevertheless, ensuing analysis results clearly indicate the interplay between the adaptation gain and the $\sigma$—modification gain in spite of the limitation mentioned above.

### A. Nominal analysis

Figure 6 shows nominal tracking responses for the adaptation gain and the $\sigma$—modification considered in our analysis. Regardless of the employed adaptive parameters, all the cases exhibit excellent tracking, which can be attributed to the fact that the system matrices in (38) are obtained from the linearization with respect to the initial condition, and therefore the linear model should be a reasonable approximation for the nonlinear model starting at the given initial condition. Since Lemma 2 also provides a measure for a tolerable norm-bounded uncertainty, in Table 1, we obtain the convergence rate for the Lyapunov function $V(t) = \zeta(t)^\top X \zeta(t)$, where $X$ is obtained by (26)
Figure 6. Time responses for $p_c, p(t)$
with varying adaptation gains and $\sigma$—modification gains. For a fixed adaptation gain, an increase in the $\sigma$—modification gain consistently increases the convergence rate in (26). However, no consistent trend with varying adaptation gains was observed with the $\sigma$—modification fixed. Since this value provides a measure for an unmatched uncertainty for which the closed-loop system remains stable, the results in Table 1 imply that increasing the $\sigma$—modification tends to increase the robustness to unmatched uncertainties. This further explains the relatively large values of $\sigma$—modification gains used in our analysis.

\[
\begin{array}{ccc}
\gamma = 0.1 & \gamma = 1 & \gamma = 10 \\
\sigma = 0.1 & 0.1989 & 0.1951 & 0.1945 \\
\sigma = 1 & 1.9988 & 1.6939 & 1.9214 \\
\sigma = 10 & 2.9268 & 2.9325 & 2.9234 \\
\end{array}
\]

Table 1. Convergence rate $\mu$ in (26)

B. Stability margin analysis

Solving the set of LMIs in (31) with varying $\gamma$ and $\sigma$ leads to the $L_2$ gain $\gamma_1$ from $w_p$ to $z_p$. Then the adaptive closed-loop system remains stable for all the uncertain mappings $\Delta(-)$, depicted in Figure 1, whose $L_2$ gain is less than $\gamma_2 = \gamma_1$.

Table 2 shows the disk size of the tolerable uncertainty with varying adaptation gains and $\sigma$—modification gains.

\[
\begin{array}{ccc}
\gamma = 0.1 & \gamma = 1 & \gamma = 10 \\
\sigma = 0.1 & 0.4779 & 0.4693 & 0.3818 \\
\sigma = 1 & 0.4774 & 0.4778 & 0.4692 \\
\sigma = 10 & 0.4776 & 0.4775 & 0.4775 \\
\end{array}
\]

Table 2. Tolerable disk size $\gamma_2$

Tables 3 and 4 show GMs and DMs obtained following the procedure in Section IV. The highest cross-over frequency among corners is 14.7288 rad/s for all the cases. Note that this frequency is increased slightly from the cross-over frequency only for the linear controller. Figure 7 shows tracking responses when the time-delays in Table 4 are introduced in the input channel.

\[
\begin{array}{ccc}
\gamma = 0.1 & \gamma = 1 & \gamma = 10 \\
GM (dec/increase)(dB) & GM (dec/increase)(dB) & GM (dec/increase)(dB) \\
\sigma = 0.1 & -3.3932/5.6456 & -3.3424/5.5035 & -2.8087/4.1771 \\
\sigma = 1 & -3.3900/5.6368 & -3.3924/5.6436 & -3.3414/5.5008 \\
\sigma = 10 & -3.3912/5.6402 & -3.3906/5.6383 & -3.3908/5.6389 \\
\end{array}
\]

Table 3. Gain margins with varying $\gamma$ and $\sigma$

The bounded tracking responses in Figure 7 verify that the time-delay margins obtained in Table 4 are guaranteed. Table 4, together with Table 2, reveals that when the $\sigma$—modification gain is fixed, increasing the adaptation gain tends to decrease the size of tolerable disk and hence decrease the time-delay margin. When the adaptation gain is fixed, in most cases, increasing the $\sigma$—modification gain tends to increase the size of the tolerable disk. Therefore, increasing the $\sigma$—modification factor tends to increase the time-delay margin.
Figure 7. Verification of guaranteed time-delays
\[
\begin{array}{|ccc|}
\hline
\gamma = 0.1 & \gamma = 1 & \gamma = 10 \\
\text{DM (sec.)} & \text{DM} & \text{DM} \\
\hline
\sigma = 0.1 & 0.0328 & 0.0322 & 0.0261 \\
\sigma = 1.0 & 0.0327 & 0.0328 & 0.0322 \\
\sigma = 10 & 0.0327 & 0.0327 & 0.0327 \\
\hline
\end{array}
\]

Table 4. DMs with varying $\gamma$ and $\sigma$

The conservatism associated with time-delay margins in Table 4 is investigated by finding the true gain variation and the time delay in Tables 5 and 6. The $GM_{\text{dec}} = -\infty$ for all the cases. These value are iteratively obtained in the GTM simulation, beyond which the adaptive control systems goes unstable. The corresponding time responses for these time delays are shown in Figure 8. It

\[
\begin{array}{|ccc|}
\hline
\gamma = 0.1 & \gamma = 1 & \gamma = 10 \\
GM_{\text{inc}}(\text{dB}) & GM_{\text{inc}}(\text{dB}) & GM_{\text{inc}}(\text{dB}) \\
\hline
\sigma = 0.1 & 30.1301 & 30.1301 & 30.1301 \\
\sigma = 1.0 & 29.9109 & 29.9084 & 29.8776 \\
\sigma = 10 & 27.8890 & 27.8539 & 27.7656 \\
\hline
\end{array}
\]

Table 5. True GMs with varying $\gamma$ and $\sigma$

is interesting that the true delay-margin is increased in comparison to the linear controller only for the case of low adaptation gain. As the adaptation gain increases, the true time-delay margin decreases. The guaranteed time-delay margins exhibits less conservatism as the adaptation gain increases.

VI. Conclusions and Future Directions

We have shown that casting the error dynamics of the tracking error and the weight estimation error into a linear parameter varying form permits the application of LMI-based analysis tools for analysis and computation of guaranteed stability margins for adaptive controllers with $\sigma$—modification. By taking the viewpoint that nonlinear systems can be parametrized by a set of linear systems, we are able to show that the stability margins defined in linear robust control theory are still valid even for nonlinear time-varying system adaptive control systems. However, since LMI analysis is formulated in the time-domain, phase margin is not a useful metric in this setting, hence time delay margin that allows for direct verification in simulations is considered. Since the stability margin analysis is based on sufficient conditions, the results are conservative.

Our future research will include extending the current formulation to perform quantitative time-delay margin analysis in the presence of an unmatched uncertainty as well as looking into how we can further reduce conservatism in the analysis.
Figure 8. Time responses with true time-delay margins
VII. Acknowledgments

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References