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Bounded Hybrid Connectivity Control of Networked Multiagent Systems

Gerardo De La Torre, Tansel Yucelen, and Eric Johnson

***Abstract*—Connectivity refers to the problem that the initial information flow between a group of agents is required to be preserved for all time in order to achieve a given set of system-level objectives. This paper focuses on a new connectivity control architecture for networked multiagent systems based on hybrid protocols. Specifically, for a group of agents subject to Δ-disk proximity graphs, we consider the consensus problem while preserving connectedness with *a priori* given bounds on the control inputs. The bounds on the control inputs are shown neither to depend on potential functions nor system states. Instead the bounds are user selected continuous functions. Using the proposed framework, we show that the maximum distance among neighboring agents strictly decreases, and hence, the connectivity problem is solved while satisfying the consensus objective. Illustrative numerical examples are provided to demonstrate the efficacy of the proposed bounded hybrid connectivity control architecture.**

***Index Terms*– Networked multiagent systems, connectivity control, hybrid control, bounded control, consensus**

I. Introduction

Networked multiagent systems consist of groups of agents that locally sense the environment, communicate with each other, and process information in order to achieve a given set of system-level objectives that are difficult for a single agent to accomplish. From a coordination and cooperation standpoint, recent decades have witnessed an increasing attention to networked multiagent systems since they are of importance to, for example, physical sciences, computer science, and control engineering communities [1]–[3].

Connectivity refers to the problem that the initial information flow between a group of agents is required to be preserved for all time in order to achieve a given set of system-level objectives and is one of the fundamental problems in networked multiagent systems. In the context of mobile robot networks, for example, this problem arises due to the fact that each robot can locally sense the environment subject to a limited sensor range [4]. Motivated from this standpoint, this paper focuses on a new connectivity control architecture based on hybrid protocols in the context of the consensus problem.

The agreement of networked agents upon certain quantities of interest is called consensus and has widespread applications in diverse areas in science and engineering. The consensus problem has known solutions under the common assumption that connectedness is maintained for all time [5]. Furthermore, it also has solutions if connectedness is assumed to be maintained only at distinct times [6]–[11]. However, such assumptions may not be valid in a networked multiagent system in general, since the information flow between an agent and its neighbours degrades as a function of distance. Losing connectivity in a network not only prevents consensus but also leads to undesired multiagent system behavior [12].

There have been a few notable approaches in the networked multiagent systems literature that solves the consensus problem while preserving connectedness. The authors in [12]–[14] propose connectivity control methodologies that are predicated on potential-type cost functions. In particular, as the distance between an agent and its neighboring agents increases, the control input of these agents also increases so that these methodologies can maintain connectivity of the overall network for all time. However, these approaches can require large control efforts especially at the critical agent-wise distance, the distance where agents are marginally connected. In the presence of actuator magnitude saturation constraints it is still possible that connectivity may be lost. Therefore, it is of practical importance to solve the connectivity control problem with bounded control inputs. The authors in [15], [16] have propose connectivity control frameworks with bounded control inputs. However, the bounds of the control inputs depend on potential functions. Maintaining fixed control bounds requires computation of potential function gradients and continuous tuning of control gains.

In this paper, we consider the consensus problem of a group of agents subject to Δ-disk proximity graphs while preserving connectedness with *a priori* given bounds on the control inputs[1]. The bounds on the control inputs are shown neither to depend on potential functions nor system states. Instead, the bounds are user selected continuous functions. Using hybrid protocols, we show that the maximum distance among neighboring agents strictly decreases, and hence, the proposed framework solves the connectivity problem while achieving the consensus objective[2]. In addition, the proposed approach is shown to have flexibility in the construction of the connectivity protocol and design guidelines are discussed. Illustrative numerical examples are provided to demonstrate the efficacy of the proposed bounded hybrid connectivity

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[1]The proposed methodology is similar in spirit to the approaches documented in [7] and [17], since we limit the set of possible inputs to preserve connectedness. However, our architecture not only preserves connectedness but also provides an arbitrary bound on the control inputs and a stability analysis.

[2]Although we consider consensus problem as a benchmark study to present the main contribution of this paper, our results can be extended to other distributed multiagent control problems (e.g., formation, cyclic pursuit, swarming, etc.) with similar assumptions and equivalent control protocol constructions.

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control architecture.

The organization of the paper is as follows. Section 2 overviews basic results from graph theory and networked multiagent systems. We present the proposed hybrid connectivity control architecture and its stability analysis in Section 3. Numerical examples are provided in Section 4 and concluding remarks are summarized in Section 5.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{R}_+$ (resp., $\overline{\mathbb{R}}_+$) denotes the set of positive (resp., nonnegative) real numbers, $\mathbb{R}_+^{n\times m}$ (resp., $\overline{\mathbb{R}}_+^{n\times m}$) denotes the set of positive-definite (resp., nonnegative-definite) $n \times m$ real matrices, $\mathbb{S}_+^{n\times n}$ (resp., $\overline{\mathbb{S}}_+^{n\times n}$) denotes the set of $n \times n$ symmetric positive-definite (resp., symmetric nonnegative-definite) real matrices, $[M]_i$ denotes the i^{th} row of matrix M, $\operatorname{diag}(v)$ denotes a diagonal matrix with scalar entries given by v, I_n denotes the $n\times n$ identity matrix, $(\cdot)^{\mathrm{T}}$ denotes the transpose, $\otimes$ denotes the Kronecker product, $\cap$ denotes the intersection, $\cup$ denotes the union, $\subseteq$ denotes the subset, $\subset$ denotes the proper (strict) subset, "$\triangleq$" denotes the equality by definition, and $\mathbf{1}_n$ denotes a $n \times 1$ vector with 1 in all entries. In addition, we write $|\cdot|$ for the absolute value or the cardinality of a set, $\|\cdot\|$ for the euclidean norm, $\max(\cdot)$ for the maximum, $\arg\max(\cdot)$ for the argument of the maximum, and $\arccos(\cdot)$ for the inverse cosine.

II. Preliminaries

In this section, we recall some of the basic notions from graph theory and networked multiagent systems[3].

A. Graphs and Their Algebraic Representation

Graphs are broadly used in networked multiagent systems to encode interactions between a group of agents. An *undirected* graph $\mathcal{G}$ is defined by a set $\mathcal{V}_\mathcal{G} = \{1, \dots, n\}$ of *nodes* and a set $\mathcal{E}_\mathcal{G} \subset \mathcal{V}_\mathcal{G} \times \mathcal{V}_\mathcal{G}$ of *edges*. If the *unordered* pair $(i, j) \in \mathcal{E}_\mathcal{G}$, then nodes i and j are *neighbors* and the neighboring relation is indicated with $i \sim j$. The set of neighbors of node i is denoted by $\mathcal{N}_\mathcal{G}(i) = \{j|(i \sim j) \in \mathcal{E}(\mathcal{G})\}$. The *degree* of a node is given by the number of its neighbors. The *degree* matrix $\Delta(\mathcal{G}) \in \mathbb{R}^{n\times n}$ of a graph $\mathcal{G}$ is given by $\Delta(\mathcal{G}) \triangleq \operatorname{diag}(d), d = [d_1, \dots, d_n]^{\mathrm{T}}$ where d_i be the degree of node i. The *adjacency* matrix $\mathcal{A}(\mathcal{G}) \in \mathbb{R}^{n\times n}$ of a graph $\mathcal{G}$ is given by

$$[A(\mathcal{G})]_{ij} \triangleq \begin{cases} 1, & \text{if } (i,j) \in \mathcal{E}_\mathcal{G}, \\ 0, & \text{otherwise.} \end{cases} \qquad (1)$$

The *Laplacian* matrix $\mathcal{L}(\mathcal{G}) \in \bar{\mathbb{S}}_+^{n\times n}$ of a graph plays a central role in many graph-theoretic treatments of networked multiagent systems and is given by

$$\mathcal{L}(\mathcal{G}) \triangleq \Delta(\mathcal{G}) - \mathcal{A}(\mathcal{G}), \qquad (2)$$

where the spectrum of the Laplacian for a connected, undirected graph can be ordered as $0 = \lambda_1(\mathcal{L}(\mathcal{G})) < \lambda_2(\mathcal{L}(\mathcal{G})) \leq \cdots \leq \lambda_n(\mathcal{L}(\mathcal{G}))$ with $\mathbf{1}_n$ as the eigenvector corresponding to the zero eigenvalue $\lambda_1(\mathcal{L}(\mathcal{G}))$ such that $\mathcal{L}(\mathcal{G})\mathbf{1}_n = \mathbf{0}_n$ holds.

[3]We refer to [2] and [18] for additional details.

A *directed* graph (or *digraph*) $\mathcal{D} = (\mathcal{V}_\mathcal{D}, \mathcal{E}_\mathcal{D})$ removes the requirement that the edge set $\mathcal{E}_\mathcal{D}$ contains unordered pairs of nodes. If the ordered pair $(i, j) \in \mathcal{E}_\mathcal{D}$ then node i is said to be the *tail* of the edge and node j is its *head*. In networked multiagent systems, an edge's orientation determines the direction of information flow between two agents.

In a *weighted* graph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G}, w)$ a function $w : \mathcal{E}_\mathcal{G} \to \mathbb{R}$ associates a value (weight) to each edge. The weighted graph Laplacian associated with the weighted digraph $\mathcal{D} = (\mathcal{V}_\mathcal{D}, \mathcal{E}_\mathcal{D}, w)$ is given as $\mathcal{L}(\mathcal{D}) = \Delta(\mathcal{D}) - \mathcal{A}(\mathcal{D})$ where

$$[A(\mathcal{D})]_{ij} \triangleq \begin{cases} w_{ij}, & \text{if } (i,j) \in \mathcal{E}_\mathcal{D}, \\ 0, & \text{otherwise,} \end{cases} \qquad (3)$$

and $[\Delta(\mathcal{D})]_{ii} = d_{\text{out}}(i), \forall i \in \mathcal{V}_\mathcal{D}$ where $d_{\text{out}}(i) = \sum_{\{j|(j,i)\in\mathcal{E}(\mathcal{D})\}} w_{ij}$ is the *weighted out-degree* of node i.

Lemma 2.1 [2]. Let $\mathcal{D}$ be a weighted digraph with n nodes. Then, the spectrum of $\mathcal{L}(\mathcal{D})$ lies in the region

$$\{z \in \mathbb{C}|\ |z - \bar{d}_{\text{out}}(\mathcal{D})| \leq \bar{d}_{\text{out}}(\mathcal{D})\}, \qquad (4)$$

where $\bar{d}_{\text{out}}$ denotes the maximum weighted out-degree in $\mathcal{D}$. That is, for every digraph $\mathcal{D}$, the eigenvalues of $\mathcal{L}(\mathcal{D})$ lie in the right-hand side of the complex plane.

B. Consensus Problem

Networked multiagent systems can be modeled by a graph $\mathcal{G}$ where nodes and edges represent agents and interagent information exchange links, respectively. In particular, let $x_i(t) \in \mathbb{R}^N$ denote the state of agent i at time $t \geq 0$ whose dynamics are described by

$$\dot{x}_i(t) = u_i(t), \quad x_i(0) = x_{i0}, \quad i = 1, \dots, n, \quad t \geq 0, \qquad (5)$$

where $u_i(t) \in \mathbb{R}^N$ denotes the control input of agent i. If agent i is allowed to access the relative state information with respect to its neighbors a solution of the consensus problem can be given by

$$u_i(t) = -\sum_{i\sim j}(x_i(t) - x_j(t)), \quad i = 1, \cdots, n, \qquad (6)$$

where connectedness of the network is required to be assumed for $t \geq 0$. The networked multiagent system given by (5) and (6) can be described as

$$\dot{x}(t) = -\big(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_N\big)x(t), \quad x(0) = x_0, \quad t \geq 0, \qquad (7)$$

where $x(t) = [x_1^{\mathrm{T}}(t), \cdots, x_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{Nn}$ denotes the aggregated state vector. Since we seek a solution to the connectedness problem, we assume that individual agent states evolve in $\mathbb{R}^3$ (i.e., $N = 3$) and in this case equation (7) is given by

$$\dot{x}(t) = -\big(\mathcal{L}(\mathcal{G}) \otimes \mathrm{I}_3\big)x(t), \quad x(0) = x_0, \quad t \geq 0. \qquad (8)$$

However, our results can be directly extended to the general case. Finally, considering (7), we note that $x(t) \to [(\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}/n)\otimes\mathrm{I}_N]x_0$ as $t \to \infty$ since the undirected graph $\mathcal{G}$ is assumed to be connected [2]. That is, the networked multiagent

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system is said to reach a consensus since $x_1 = x_2 = \cdots = x_n$ holds.

III. Hybrid Connectivity Control

A. Proposed Architecture

Consider a system of n agents exchanging relative state information according to a connected, undirected graph $\mathcal{G}$. Let $x(t) = [x_1(t), \cdots, x_n(t)]^{\mathrm{T}} \in \mathbb{R}^{3n}$ and $u(t)=[u_1(t), \cdots, u_n(t)]^{\mathrm{T}} \in \mathbb{R}^{3n}$ denote the aggregated state vector and control input of the networked multiagent system, respectively. Furthermore, let the state of agent i evolve according to (5). Suppose, the graph $\mathcal{G}$ is subjected to a Δ-disk proximity graph [2] such that $\mathcal{G}(t) = (\mathcal{V}_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}}(t))$ where $(i, j) \notin \mathcal{E}_{\mathcal{G}}(t)$ if $\|x_i(t) - x_j(t)\| > \Delta$ and $\Delta \in \mathbb{R}_+$. Therefore, a pair of agents cannot exchange information if the distance between them is sufficiently large. However, this does not imply that if $\|x_i(t) - x_j(t)\| \leq \Delta$ then $(i, j) \in \mathcal{E}_{\mathcal{G}}(t)$.

To address the connectivity problem of networked multiagent systems we propose the hybrid consensus protocol given by

$$u_i(t) = \underset{v \in U_i \cup 0}{\arg\max} \left(v^{\mathrm{T}} \sum_{j^\star \in M_i} \left(x_{j^\star}(t) - x_i(t)\right) \right), \quad (9)$$

$$U_i(t) \triangleq \bigcap_{j^\star \in M_i} S^i_{j^\star}(t), \quad (10)$$

$$S^i_{j^\star}(t) \triangleq \{y \in \mathbb{R}^3 : \|y\| \leq \alpha_i(t), \; \frac{y^{\mathrm{T}}(x_{j^\star}(t) - x_i(t))}{\|y\|\|x_{j^\star}(t) - x_i(t)\|} > 0, j^\star \in M_i\}, \quad (11)$$

$$M_i(t) \triangleq \{\underset{j \in \mathcal{N}_{\mathcal{G}}(i)}{\arg\max}(\|x_j(t) - x_i(t)\|)\}. \quad (12)$$

where $\alpha_i(t) : \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ is a continuous positive definite function. $M_i(t) \subseteq \mathcal{N}_{\mathcal{G}}(i)$ contains all neighboring agents that are the furthest away from agent i. Note if $y \sim z$ then $z \in M_y$ does not imply $y \in M_z$. Figure 1 illustrates the vectors contained in $S^i_{j^\star}$. Notice all vectors in $S^i_{j^\star}$ are bounded by $\alpha_i(t)$ and the angle between all vectors in $S^i_{j^\star}$ and the vector from agent i to $j^\star$ is less than 90 degrees. $S^i_{j^\star}$ contains all local control inputs that do not increase $\|x_{j^\star}(t) - x_i(t)\|$ where $j^\star \in M_i$. The selected local input, u_i, is common among all $S^i_{j^\star}$ (or 0) and $u_i^{\mathrm{T}} \sum_{j^\star \in M_i} \left(x_{j^\star}(t) - x_i(t)\right)$ is maximized. Effectively, the control architecture limits possible local control inputs to those that will not increase $\max_{j \in \mathcal{N}_{\mathcal{G}}(i)}(\|x_j(t) - x_i(t)\|)$. Finally, note that the control bounds can be made agent specific and only relative local information is used. Therefore, the proposed consensus protocol is decentralized.

B. Preserving Connectedness and Achieving Consensus

To show that the proposed hybrid connectivity control architecture preserves connectedness and solves the consensus problem the following results are required. It should be noted that Lemmas 3.1 to 3.5 assume that the graph is connected.

Lemma 3.1. The set $U_i(t)$ is non-empty if and only if there exist an *open* halfspace, $\mathcal{H}_i \subset \mathbb{R}^3$, created by a plane, $\mathcal{P}_i$, such that $x_{j^\star}(t) \in \mathcal{H}_i$, $\forall j^\star \in M_i(t)$ and $x_i(t) \in \mathcal{P}_i$.

Proof. To prove sufficiency note that there exist $v_i \in \mathbb{R}^3$ such that $v_i - x_i \perp \mathcal{P}_i$ and $v_i \in \mathcal{H}_i$. Furthermore,

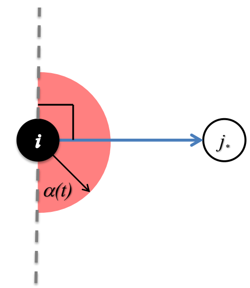

Fig. 1: Illustration of the set $S^i_{j^\star}$

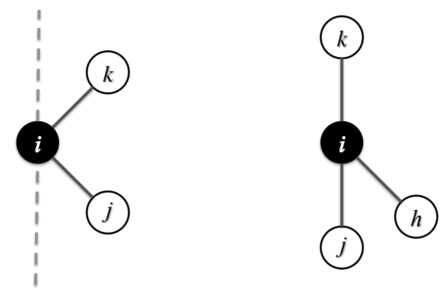

Fig. 2: Illustration of the result in Lemma 3.1

$\frac{(v_i - x_i)^{\mathrm{T}}(z - x_i)}{\|v_i - x_i\|\|z - x_i\|} > 0$ for every $z \in \mathcal{H}_i$. Now, it follows that $\frac{(v_i - x_i)^{\mathrm{T}}(x_{j^\star} - x_i)}{\|v_i - x_i\|\|x_{j^\star} - x_i\|} > 0$ for all $j^\star \in M_i$ and $v_i - x_i \in U_i$.

To show necessity, if U_i is non-empty, then there exist $w_i \in \mathbb{R}^3$ such that $\|w_i - x_i\| \leq \alpha_i(t)$ and $\arccos\left(\frac{(w_i - x_i)^{\mathrm{T}}(x^\star_j - x_i)}{\|w_i - x_i\|\|x^\star_j - x_i\|}\right) < \frac{\pi}{2}$, $j^\star \in M_i$. Now, it follows that $\arccos\left(\frac{(x^\star_k - x_i)^{\mathrm{T}}(x^\star_j - x_i)}{\|x^\star_k - x_i\|\|x^\star_j - x_i\|}\right) < \pi$ for all $j^\star, k^\star \in M_i$. Therefore, $\mathcal{H}_i$ and $\mathcal{P}_i$ exist. □

The result in Lemma 3.1 can be easily seen by examining Figure 2. For the system on the left, the set U_i is non-empty since $\mathcal{H}_i$ exists, but for the system on the right, no such open halfspace exists. The existence of $\mathcal{H}_i$ ensures that agent i can move infinitesimally into $\mathcal{H}_i$ such that $\max_{j \in \mathcal{N}_{\mathcal{G}}(i)}(\|x_j(t) - x_i(t)\|)$ does not increase (if the other agents are fixed). Therefore, in general, the existence of such a halfspace $\mathcal{H}_i$ ensures that U_i is non-empty.

Lemma 3.2. The local control input $u_i = 0$ if and only if U_i is empty.

Proof. For necessity, assume that $u_i = 0$ and U_i is non-empty. Lemma 3.1 shows that $v_i - x_i \in U_i$ and $\frac{(v_i - x_i)^{\mathrm{T}}(z - x_i)}{\|v_i - x_i\|\|z - x_i\|} > 0$, $z \in \mathcal{H}_i$. In addition, since U_i is non-empty $x_{j^\star}(t) \in \mathcal{H}_i$ for all $j^\star \in M_i$. Since $(v_i - x_i)^{\mathrm{T}} \sum_{j^\star \in M_i}(x_{j^\star}(t) - x_i(t)) > 0$ we arrive at a contradiction since $u_i \neq 0$. To show sufficiency, if U_i is empty, then $U_i \cup 0 = 0$, and as a direct consequence, $u_i = 0$. □

Remark 3.1. U_i is empty if $x(t) = \mathbf{1}_n \otimes z$ for some $z \in \mathbb{R}^3$ (i.e. if agents reach consensus). In other words, when consensus is reached, the set $S^i_{j^\star}$ must be empty since $\|x_{j^\star}(t) - x_i(t)\| = 0$ for all $j^\star \in M_i$.

Lemma 3.3. The control input $u_i(t)$ can be parameterized as $u_i(t) = \bar{w}_i(t) \sum_{j^\star \in M_i}(x_{j^\star}(t) - x_i(t))$ where $\bar{w}_i(t) \in \bar{\mathbb{R}}_+$.

Proof. First notice that maximizing $u_i^{\mathrm{T}} \sum_{j^\star \in M_i}(x_{j^\star}(t) -$

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$x_i(t)\big)$ is equivalent to maximizing $\|u_i\|\|\sum_{j^\star\in M_i}\big(x_{j^\star}(t)-x_i(t)\big)\|cos(\theta_u)$ and that the maximum is given when $\|u_i\|=\alpha_i(t)$ and $\theta_u=0$[4]. If $U_i(t)$ is non-empty then, from Lemma 3.1, $\frac{(x_k^\star-x_i)^{\mathrm{T}}(x_j^\star-x_i)}{\|x_k^\star-x_i\|\|x_j^\star-x_i\|}>0$ for all $j^\star,k^\star\in M_i$. It follows that $\frac{\sum_{k^\star\in M_i}\big(x_{k^\star}(t)-x_i(t)\big)^{\mathrm{T}}(x_{j^\star}(t)-x_i(t))}{\|\sum_{k^\star\in M_i}\big(x_{k^\star}(t)-x_i(t)\big)\|\|x_{j^\star}(t)-x_i(t)\|}>0, j^\star\in M_i$. Thus, $u_i(t)=\alpha_i(t)\frac{\sum_{j^\star\in M_i}\big(x_{j^\star}(t)-x_i(t)\big)}{\|\sum_{j^\star\in M_i}\big(x_{j^\star}(t)-x_i(t)\big)\|}$ and $\bar{w}_i(t)=\alpha_i(t)\|\sum_{j^\star\in M_i}\big(x_{j^\star}(t)-x_i(t)\big)\|^{-1}$. If $U_i(t)$ is empty then $u_i(t)=0$ and $\bar{w}_i(t)=0$. □

Lemma 3.3 shows that the static undirected graph given by (8) becomes a dynamic directed graph with time varying edge weights when the proposed architecture is used. Rather than considering each neighbor equally, agents simply *ignore* all neighbors besides those that are furthest away. As the multiagent system evolves, agents may begin to *ignore* different neighbors causing a change in the system's graph. The multiagent system can now be described by

$$\dot{x}(t)=-\mathcal{W}(t)\mathcal{L}(\mathcal{D}(t))x(t),\quad x(0)=x_0,\quad t\geq 0,\tag{13}$$

that is, by a time-varying dynamic digraph given by $\mathcal{D}(t)=(\mathcal{V}_\mathcal{D},\mathcal{E}_\mathcal{D}(t))$ where $\mathcal{V}_\mathcal{D}=\mathcal{V}_\mathcal{G}$, the ordered pair $(i,j)\in\mathcal{E}_\mathcal{D}(t)$ only if $j\in M_i(t)$ and $\mathcal{W}(t)=\mathrm{diag}[\bar{w}_1(t),\ldots,\bar{w}_n(t)]$. Notice that $[-\mathcal{W}(t)\mathcal{L}(\mathcal{D}(t))x(t)]_i=\bar{w}_i(t)\sum_{j^\star\in M_i}(x_{j^\star}(t)-x_i(t))$.

Lemma 3.4. $\mathcal{L}(\mathcal{D}(t))x=0$ if and only if $x(t)=\mathbf{1}_n\otimes z$ for some $z\in\mathbb{R}^3$ for all $\mathcal{E}_\mathcal{D}(t)$.

Proof. Sufficiency is immediate from the results in Lemmas 3.1 and 3.2. For necessity, if $x(t)\neq\mathbf{1}_n\otimes z$, then there exist an agent s such that $\|x_s(t)-\frac{1}{n}\sum_{j\in\mathcal{V}_\mathcal{G}}x_j(t)\|\geq\|x_i(t)-\frac{1}{n}\sum_{j\in\mathcal{V}_\mathcal{G}}x_j(t)\|$, $i\in\mathcal{V}_\mathcal{G}$. We can construct an open halfspace, $\mathcal{H}_s$, by creating a plane, $\mathcal{P}_s$, such that $x_s(t)-\frac{1}{n}\sum_{j\in\mathcal{V}_\mathcal{G}}x_j(t)\perp\mathcal{P}_s$ and $\frac{1}{n}\sum_{j\in\mathcal{V}_\mathcal{G}}x_j(t)\in\mathcal{H}_s$. Since the distance from the system's centroid to agent s is maximum among all agents, we can conclude that $x_{j^\star}(t)\in\mathcal{H}_s$ for all $j^\star\in M_s\subset\mathcal{V}_\mathcal{G}$. From Lemmas 3.1 and 3.2, it follows that U_s is non-empty and $u_s\neq 0$. As a direct consequence, $\mathcal{L}(\mathcal{D}(t))x\neq 0$ for all $\mathcal{E}_\mathcal{D}(t)$. □

Figure 3 shows how the system centroid is used to construct $\mathcal{H}_s$. Note that since agent s is furthest from the centroid $\mathcal{H}_s$ is guaranteed to exist. Lemma 3.4 proves that the proposed architecture does not induce additional equilibrium points to the system.

The following results make use of a subgraph $\mathcal{G}_\mathcal{R}(t)=(\mathcal{V}_{\mathcal{G}_\mathcal{R}}(t),\mathcal{E}_{\mathcal{G}_\mathcal{R}}(t))\subseteq\mathcal{G}$ given by

$$\mathcal{V}_{\mathcal{G}_\mathcal{R}}(t)=\{i\in\mathcal{V}_\mathcal{G}:\arg\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i-x_j\|\}\tag{14}$$

$$\mathcal{E}_{\mathcal{G}_\mathcal{R}}(t)=\{(i,j):(i,j)\in\mathcal{E}_\mathcal{G},\ i,j\in\mathcal{V}_{\mathcal{G}_\mathcal{R}}\}.\tag{15}$$

In particular, $\mathcal{G}_\mathcal{R}$ is a subgraph of $\mathcal{G}$ such that all distances between all neighbors in $\mathcal{G}_\mathcal{R}$ is the maximum distance between all neighbors in $\mathcal{G}$. Notice that $\mathcal{G}_\mathcal{R}$ is guaranteed to contain at least a single edge and two nodes. If the maximum distance between neighbors is not uniquely defined then $\mathcal{G}_\mathcal{R}$ will contain more than one edge. Furthermore, notice that there exist

[4]Where $\theta_u=\arccos\Big(\frac{u_i^{\mathrm{T}}\sum_{j^\star\in M_i}\big(x_{j^\star}(t)-x_i(t)\big)}{\|u_i\|\|\sum_{j^\star\in M_i}\big(x_{j^\star}(t)-x_i(t)\big)\|}\Big)$

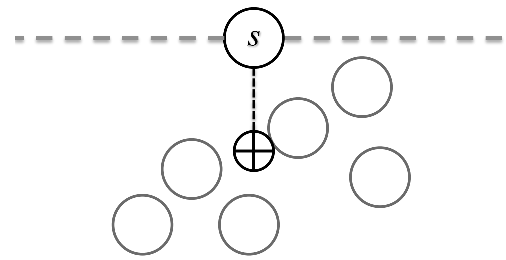

Fig. 3: Illustration of the result in Lemma 3.4

a finite number of possible $\mathcal{V}_{\mathcal{G}_\mathcal{R}}(t)$ and $\mathcal{E}_{\mathcal{G}_\mathcal{R}}(t)$ and therefore $\mathcal{V}_{\mathcal{G}_\mathcal{R}}(t)=\mathcal{V}_{\mathcal{G}_\mathcal{R}^k}$, $\mathcal{E}_{\mathcal{G}_\mathcal{R}}(t)=\mathcal{E}_{\mathcal{G}_\mathcal{R}^k}$, and $\mathcal{G}_\mathcal{R}(t)=\mathcal{G}_\mathcal{R}^k\in\bar{\mathcal{G}}_\mathcal{R}$, $k\in\{1,\ldots,p\}$ where $\bar{\mathcal{G}}_\mathcal{R}$ is a finite set of subgraphs. The dynamics of the networked agents of the subgraph can be described as

$$\dot{x}_\mathcal{R}(t)=-\big(\mathcal{W}_\mathcal{R}(t)\mathcal{L}(\mathcal{G}_\mathcal{R}^k)\otimes\mathrm{I}_3\big)x_\mathcal{R}(t),\ x_\mathcal{R}(0)=x_{\mathcal{R}0}\tag{16}$$

for $k\in\{1,\ldots,p\}$ and $t\geq 0$, where $x_\mathcal{R}(t)=[x_1(t),\ldots,x_{|\mathcal{V}_{\mathcal{G}_\mathcal{R}}|}(t)]^{\mathrm{T}}$ denotes the aggregated state vector of the agents of the subgraph, $\mathcal{W}_\mathcal{R}(t)=\mathrm{diag}([\bar{w}_1(t),\ldots,\bar{w}_{|\mathcal{V}_{\mathcal{G}_\mathcal{R}}|}(t)])$ denotes the respective input weights of the agents in the subgraph, and $\mathcal{L}(\mathcal{G}_\mathcal{R}^k)$ is the *unweighted* graph Laplacian induced by the subgraph $\mathcal{G}_\mathcal{R}^k$. Therefore, the state dependent subgraph given by $\mathcal{G}_\mathcal{R}=(\mathcal{V}_{\mathcal{G}_\mathcal{R}},\mathcal{E}_{\mathcal{G}_\mathcal{R}})$ produces a *switched system* with a state dependent *switching signal* since the graph's topology is switching. Notice that this imples that (13) is also a switched system with finite subsystems. For the remainder of this note it is assumed that there is not infinite number of switches in any finite time interval. This assumption is standard for systems with switching topologies [19].

Remark 3.2. In order to further justify the assumption made above the definition of $M_i(t)$ can be modified to incorporate a small dwell time. Specifically, if $\|x_i(\tau)-x_j(\tau)\|\geq\|x_i(\tau)-x_k(\tau)\|$, $\forall k\in\mathcal{N}_\mathcal{G}(i)$ then $j\in M_i(t),\tau\leq t\leq\tau+\epsilon$ for some small $\epsilon>0$. Notice that $M_i(t)$ still contains all neighbors that are furthest away from agent i at time t, but now also contains those agents that were recently furthest away. Therefore, the control architecture will still limit possible local control inputs to those that will not increase $\max_{j\in\mathcal{N}_\mathcal{G}(i)}(\|x_j(t)-x_i(t)\|)$. Although our results can be readily extended to this case, we do not consider this modification in order to concentrate on the main contributions of this paper and for ease of exposition.

Lemma 3.5. $\sum_{i\in\mathcal{V}_{\mathcal{G}_\mathcal{R}}^k}|u_i|\neq 0$ if $x(t)\neq\mathbf{1}_n\otimes z$ for some $z\in\mathbb{R}^3$ and $\forall k\in\{1,\ldots,p\}$.

Proof. The proof follows directly from the proof of Lemma 3.4 by considering the subgraph $\mathcal{G}_\mathcal{R}$ instead of the graph $\mathcal{G}$.

The following theorem now presents the main result of the paper.

Theorem 3.1. Consider an initially connected networked multiagent system where agents evolve according to (5) subject to the hybrid control architecture given by (9)-(12) and to a Δ-disk proximity graph. Then, the closed-loop system is Lyapunov stable for all $x_0\in\mathbb{R}^{3n}$, $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i-x_j\|$ is a strictly decreasing function, and $x(t)\to\mathbf{1}_n\otimes z$ as $t\to\infty$ for

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some $z \in \mathbb{R}^3$. Furthermore, the system will remain connected for all time.

Proof. Consider the Lyapunov candidate function given by $V(x(t)) = \frac{1}{|\mathcal{E}_{\mathcal{G}^k_\mathcal{R}}|} \sum_{(i,j)\in\mathcal{E}_{\mathcal{G}^k_\mathcal{R}}} \frac{1}{2}(x_i - x_j)^{\mathrm{T}}(x_i - x_j)$ where $k \in \{1,\ldots,p\}$ and the k^{th} subsystem is active the subsystem of the switched system given by (16). Note that $V(x(t_s^-)) = V(x(t_s^+))$ for all switching times t_s since $V(x(t))$ gives the maximum distance between neighbors on graph $\mathcal{G}$ at time t. Differentiating $V(x)$ inbetween switches along trajectories of (16) yields $\dot{V}(x(t)) = \frac{1}{|\mathcal{E}_{\mathcal{G}^k_\mathcal{R}}|}\sum_{i\in\mathcal{V}_{\mathcal{G}^k_\mathcal{R}}}\sum_{j\in M_i}(x_i - x_j)^{\mathrm{T}}u_i$. From (9)-(11) we see that $(x_j - x_i)^{\mathrm{T}}u_i > 0$ if $u_i \neq 0$ for all $(i,j) \in \mathcal{E}_{\mathcal{G}_\mathcal{R}}$, and hence, $\dot{V}(x(t)) \leq 0$. Furthermore, from Lemma 3.5 we know $\sum_{i\in\mathcal{V}^k_{\mathcal{G}_\mathcal{R}}}|u_i| \neq 0$ if $V(x(t)) \neq 0$. We can then conclude that $\dot{V}(x) < 0$ when $V(x(t)) > 0$ and $\dot{V}(x) = 0$ when $V(x(t)) = 0$. From Lemma 3.3, $\dot{V}(t) = \frac{1}{|\mathcal{E}_{\mathcal{G}^k_\mathcal{R}}|}\sum_{i\in\mathcal{V}_{\mathcal{G}^k_\mathcal{R}}}\sum_{j\in M_i}(x_i - x_j)^{\mathrm{T}}u_i$ is continuous between switches. Therefore, $V(x(t))$ is continuous and strictly decreasing when $V(x(t)) > 0$. As a result, the closed-loop system is Lyapunov stable for all $x_0 \in \mathbb{R}^{3n}$ and $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i - x_j\|$ is a strictly decreasing function.

Finally, $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i - x_j\|$ being a strictly decreasing function implies $x(t) \to \mathbf{1}_n \otimes z$ as $t \to \infty$ for some $z \in \mathbb{R}^3$ and further implies that the system will remain connected for all time. □

Remark 3.3. Notice that the hybrid control architecture given by (9)-(12) can be simplified to

$$u_i(t) = \alpha_i(t)\frac{\sum_{j^\star\in M_i}\big(x_{j^\star}(t) - x_i(t)\big)}{\|\sum_{j^\star\in M_i}\big(x_{j^\star}(t) - x_i(t)\big)\|}, \quad (17)$$

$$M_i(t) \triangleq \{\arg\max_{j\in\mathcal{N}_\mathcal{G}(i)}(\|x_j(t) - x_i(t)\|)\}, \quad (18)$$

where $\alpha_i(t) : \bar{\mathbb{R}}_+ \to \mathbb{R}_+$ is a continuous positive definite function. Therefore, implementation of the proposed consensus protocol is not computationally expensive.

Remark 3.4. The bounds on the control inputs are only required to be continuous and positive definite and are not dependent on potential functions or system states. This feature separates our approach from other approaches found in the literature, where inputs cannot be arbitrarily bounded or depend on potential functions [12]–[16]. The input bounds can be used to, for example, converse battery life or account for input saturation constraints. We illustrate a design based on this bound in the second example of the next section.

Remark 3.5. Like the definition of set $S^i_{j^\star}(t)$, the method of selecting u_i is not unique. In the presented design the input u_i is selected such that $u_i^{\mathrm{T}}\sum_{j^\star\in M_i}\big(x_{j^\star}(t) - x_i(t)\big)$ is maximized. This method is arbitrary and was selected simply as a proof of concept. As long the method allows $\sum_{i\in\mathcal{V}_{\mathcal{G}^k_\mathcal{R}}}\sum_{j\in M_i}(x_i - x_j)^{\mathrm{T}}u_i$ to be piecewise continuous and satisfies Lemma 3.2 and 3.3, the control architecture will still achieve consensus with connectedness guarantees. Thus, the proposed architecture can be customized for a specific application or situation. Furthermore, the method of selecting u_i does not need to remain the same for the entire evolution of the system. This can be used to, for example, optimize control inputs.

IV. Numerical Examples

This section illustrates the proposed hybrid connectivity control architecture given by (9)-(12) on a system that has been shown to lose connectedness when the nominal consensus input (6) is used [12]. For the following results, we assume $\Delta = 4$ and $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i(0) - x_j(0)\| = 4$. That is, the connection between (at least) one pair of neighboring agents is on the verge of being lost.

A. Example 1

For the first case, the formulation described in (9)-(12) was used. As shown in Figures 4-5, agents come to consensus. Note that the system inputs remain bounded even though $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i - x_j\| = \Delta$ at $t = 0$. Furthermore, as shown in Figure 5, $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i - x_j\|$ is strictly decreasing. By examining Figure 4 at $t = 0$ it can be seen that local inputs u_i are chosen such that $\max_{j\in\mathcal{N}_\mathcal{G}(i)}(\|x_j(t) - x_i(t)\|)$ is not increasing. Hence, by limiting possible local inputs in a distributed fashion, the system is able achieve a global objective. △

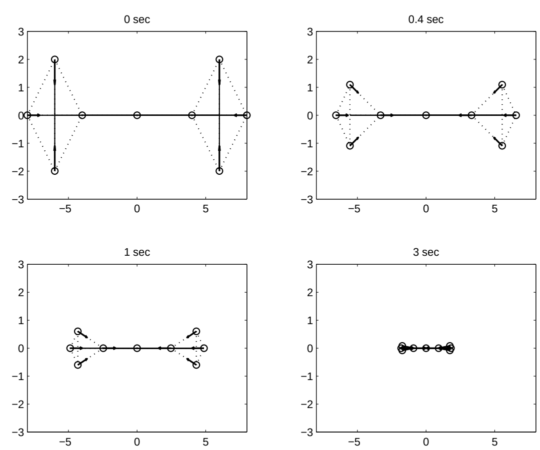

Fig. 4: Connecting solid lines indicate $\{i,j\} \in \mathcal{E}_{\mathcal{G}_\mathcal{R}}$ and solid arrows represent normalized input vectors of agents (Example 1).

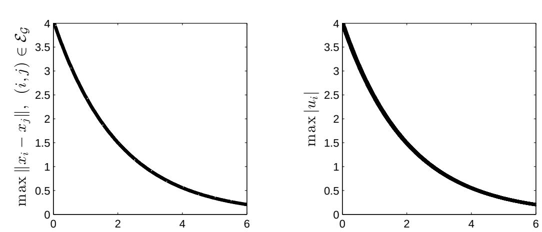

Fig. 5: Maximum interagent distance (left) and input (right) versus time (Example 1).

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B. Example 2

We now restricted the control inputs of agents to be bounded by $\alpha_i(t) = 2e^{-\frac{t}{4}}$ for all agents. As shown in Figures 6-7, agents come to consensus. As expected, the system converges at a slower rate since the control input is bounded by a decaying exponential function. However, the time-varying bound was satisfied and $\max_{(i,j)\in\mathcal{E}_\mathcal{G}}\|x_i - x_j\|$ is still strictly decreasing as expected by the proposed theory. This example highlights this key feature of our proposed approach. As discussed in Remarks 3.4 and 3.5, the proposed architecture has great flexibility in the protocol design and the architecture described by (9)-(12) is not unique in this regard. The proposed architecture is well suited to be applied to a wide range of problem and situations. $\triangle$

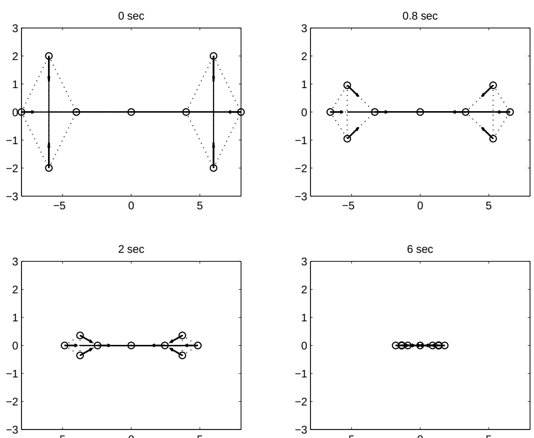

Fig. 6: Connecting solid lines indicate $\{i, j\} \in \mathcal{E}_{\mathcal{G}_\mathcal{R}}$ and solid arrows represent normalized input vectors of agents (Example 2).

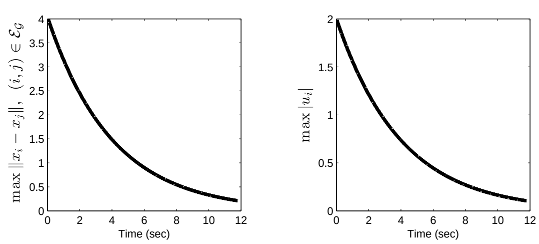

Fig. 7: Maximum interagent distance (left) and input (right) versus time (Example 2).

V. Conclusion

This paper presented a new connectivity control architecture based on hybrid protocols to preserve agent-wise connectedness. Given a group of agents subject to Δ-disk proximity graphs and *a priori* given bounds on their control inputs, we showed that the proposed architecture solves the consensus problem via strictly decreasing the maximum distance among neighboring agents. Future research will include, for example, extensions to formation and flocking problems and networked agents having high-order dynamics.

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