Consensus Protocols for Networked Multiagent Systems with Relative Position and Neighboring Velocity Information

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Abstract—The consensus problem appears frequently in the coordination of multiagent systems in science and engineering and involves the agreement of networked agents upon certain quantities of interest. In this paper, we focus on a new consensus protocol for networked multiagent systems. The proposed control protocol consists of a standard term capturing relative position information and a new term capturing neighboring velocity information. In particular, the addition of the latter term results in an increase of the rate of system convergence while maintaining a fixed graph structure and without increasing the maximum eigenvalue of the graph Laplacian. Furthermore, in certain cases, it is shown that the maximum singular value of the graph Laplacian is not increased. This departs from the traditional view that the Fiedler eigenvalue, a function of graph structure, governs the system’s rate of convergence. In addition, it is shown that a connected and undirected graph topology acts as a weighted complete graph topology with the addition of this latter term to a standard consensus protocol. A comparative numerical example is provided to demonstrate the advantages of this new consensus protocol.

I. INTRODUCTION

Networked multiagent systems consists of a group of agents that locally sense the environment, communicate with each other, and process information in order to achieve a given set of system-level objectives. Since these systems have widespread applications in physical, biological, social, economic, and engineering systems, it is not surprising that the last decade has witnessed an increased interest in networked multiagent systems. One important multiagent system behavior that has attracted much attention is consensus.

Consensus refers to the agreement of networked agents upon certain quantities of interest. Several solutions to achieve consensus exist in the literature [1]–[3]. Consensus protocols are used in networked multiagent systems to obtain system-wide behaviors such as rendezvous [4], [5], formation control [6], [7], and flocking [8]–[10]. In this paper, we focus on a new consensus protocol for networked multiagent systems. Specifically, the proposed control protocol of this paper modifies a standard consensus protocol by a new term capturing neighboring velocity information. In particular, the addition of this term results in an increase of the rate of system convergence while maintaining a fixed graph structure and without increasing the maximum eigenvalue of the graph Laplacian (maximum singular value in some cases). This departs from the traditional view that the Fiedler eigenvalue, the minimum nonzero eigenvalue of the graph Laplacian, governs the system’s rate of convergence. In addition, we showed that a multiagent system represented by a connected and undirected graph topology acts as a multiagent system represented by a weighted complete graph topology with the addition of this term to a standard consensus protocol.

The organization of the paper is as follows. Section II reviews the basic results from graph theory and networked multiagent systems that are necessary to develop the main results of our paper. We introduce and analyze the proposed consensus control protocol in Section III. Section IV provides a comparative numerical example to demonstrate the advantages of our approach versus standard control protocols. Finally, concluding remarks are summarized in Section V.

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ real matrices, $\mathbb{S}_+$ denotes the set of $n \times n$ symmetric nonnegative-definite real matrices, $\text{diag}(v)$ denotes a diagonal matrix with scalar entries given by $v$, $I_n$ denotes the $n \times n$ identity matrix, $\otimes$ denotes the Kronecker product, $\cup$ denotes the union, $\subset$ denotes the subset, “$\sim$” denotes the equality by definition, $0_n$ denotes a $n \times 1$ vector with 0 in all entries, and $1_n$ denotes a $n \times 1$ vector with 1 in all entries. In addition, we write $|\cdot|$ for the absolute value, $\lambda_i(N)$ for the $i$th eigenvalue of matrix $N$, $\sigma_i(N)$ for the $i$th singular value of matrix $N$, $(N)^{2}$ for the square root of positive-definite matrix $N$, $(N)^{-1}$ for the inverse of nonsingular matrix $N$ and $[N]_{ij}$ for the entry in the $i$th row and $j$th column of matrix $N$.

II. PRELIMINARIES

In this section, we recall some of the basic notions from graph theory and networked multiagent system\(^1\) and introduce terminology used throughout the paper.

A. Graphs and Their Algebraic Representation

Graphs are broadly used in networked multiagent systems to encode interactions between a group of agents. An undirected graph $G$ is defined by a set $V_G = \{1, \ldots, n\}$ of nodes and a set $E_G \subseteq V_G \times V_G$ of edges. If the unordered pair $(i, j) \in E_G$, then nodes $i$ and $j$ are neighbors and the neighboring relation is indicated with $i \sim j$. The set of neighbors of node $i$ is denoted by $N_G(i) = \{j \mid i \sim j\}$.

\(^1\)We refer to [11]–[13] for additional details.
If there exist a sequence of \( m \) nodes \( v_{i0}, v_{i1}, \ldots, v_{ik} \) such that nodes \( v_{ia} \) and \( v_{ia+1} \) are adjacent then a walk of length \( m \) is said to exist between nodes \( v_{i0} \) and \( v_{ik} \). In this paper we define the weight of a walk sequence as \( W_{i-k}^{m,s} (A^w(G)) = w_{i0i1} w_{i1i2} \cdots w_{i_{k-1}k} \) where the sequence \( v_{i0}, v_{i1}, \ldots, v_{ik} \) specifies the \( s \)th walk of length \( m \) starting at node \( i \) and ending at node \( k \). If \( m = 0 \) then \( W_{i-k}^{0,s} (A^w(G)) = 1 \) where \( s \in \{1\} \).

**Lemma 2.2.** [13] Let \( A(G) \) be the adjacency matrix of a graph given by \( 1 \) then \( |A(G)|_{ik} \) is the number of walks of length \( m \) starting at node \( i \) and ending at node \( k \).

**Lemma 2.3.** Let \( A^w(G) \) be the weighted adjacency matrix of a graph given by \( 6 \) then \( |A^w(G)|_{ik}^{m} = \sum_{s=1}^{m} A^w(G)_{ik} W_{i-k}^{m,s} (A^w(G)) \) where \( A^w(G) \) is the unweighted adjacency matrix of the graph.

**Proof.** This follows from a simple extension of Lemma 2.2. \( \Box \)

### B. Consensus Problem

Networked multiagent systems can be modeled by a graph \( G \), where nodes and edges, represent agents and interagent information exchange links, respectively. In particular, let \( x_{i}(t) \in \mathbb{R}^{N} \) denote the state of agent \( i \) at time \( t \geq 0 \), whose dynamics are described by

\[
\dot{x}_{i}(t) = u_{i}(t), \quad x_{i}(0) = x_{i0}, \quad i = 1, \cdots, n,
\]

with \( u_{i}(t) \in \mathbb{R}^{n} \) being the control input of agent \( i \). If agent \( i \) is allowed to access the relative state information with respect to its neighbors, a solution of the consensus problem can be given by

\[
u_{i}(t) = -\sum_{j \neq i} (x_{i}(t) - x_{j}(t)), \quad i = 1, \cdots, n,
\]

where connectedness of the network is required to be assumed for \( t \geq 0 \). The networked multiagent system given by \( 8 \) and \( 9 \) can be described as

\[
\dot{x}(t) = -L(G) \otimes I_{N} x(t), \quad x(0) = x_{0},
\]

where \( x(t) = [x_{1}^{T}(t), \cdots, x_{n}^{T}(t)]^{T} \) denotes the aggregated state vector. For ease of exposition, this paper considers the case \( N = 1 \). However, all results presented in this paper can be trivially extended to the general case. Considering \( 10 \), we note that \( x(t) \rightarrow \{[1_{n} I_{n}^{F}/n] \otimes I_{N} \} x_{0} \) as \( t \rightarrow \infty \) since the undirected graph \( G \) is assumed to be connected \([11]\).

That is, the networked multiagent system is said to reach a consensus since \( x_{1} = x_{2} = \cdots = x_{n} \) holds.

The rate of convergence of the consensus protocol is governed by \( \lambda_{2}(L(G)) \). Therefore, \( \lambda_{2}(L(G)) \), also known as the Fiedler eigenvalue, gives an important performance measure of the system. We defined the condition number of a graph Laplacian as \( \kappa(L(G)) = \sigma_{1}(L(G)) / \sigma_{2}(L(G)) \) where \( \sigma_{2}(\cdot) \leq \sigma_{3}(\cdot) \cdots \leq \sigma_{n}(\cdot) \) since \( \sigma_{1}(\cdot) = 0 \) \([14]\). If \( L(G) \) is a normal matrix then \( \kappa(L(G)) = \lambda_{2}(L(G)) / \sigma_{2}(L(G)) \). In this case, the condition number of a graph Laplacian relates the rate of convergence to the maximum singular value of the Laplacian of the system. This condition number provides an objective method for evaluating the convergence rate of
a system. Recall that high control gains can be used to increase the convergence rate. However, high gains increase the maximum singular value and eigenvalue of the system and may result in, for example, violation of control saturation limitations.

III. Consensus Protocols with Neighboring Velocity Information

In this section, we introduce and analyze a consensus protocol that consists of not only a standard term capturing relative position information but also a new term capturing neighboring velocity information. While this implies that an agent \( i \) needs access to the velocity information of its neighbors, i.e., \( \sum_{j \sim i} \dot{x}_j(t) \), as we see later, the proposed consensus protocol provides significant advantages over standard protocols composed only using relative state information as an agent in its neighbors.

Consider a system of \( n \) agents exchanging relative state information in a connected, undirected graph \( \mathcal{G} \). Let \( x(t) = [x_1(t), \ldots, x_n(t)]^T \in \mathbb{R}^n \) and \( u(t) = [u_1(t), \ldots, u_n(t)]^T \in \mathbb{R}^n \) denote the aggregated state vector and control input of the networked multiagent system, respectively. Furthermore, let the state of agent \( i \) evolve according to (8). The standard consensus protocol (9) can be augmented to include the additional neighboring velocity information as

\[
\dot{u}_i(t) = - \sum_{i \sim j} (x_i(t) - x_j(t)) + \gamma \sum_{i \sim j} \dot{x}_j(t),
\]

where \( \gamma \in (0, 1/\bar{d}_i) \) and \( \bar{d}_i \) is the maximum degree in \( \mathcal{G} \).

Remark 3.1. Agent \( i \) can obtain \( \sum_{j \sim i} \dot{x}_j(t) \) in several different ways. If agents are exchanging information locally, additional information can be shared in order to acquire this value if possible. Additionally, if agents are using proximity sensors to obtain relative state information then agents can obtain this additional information through differentiation and use of local models. In other words, agent \( i \) has access to \( \dot{x}_i(t) \) since its input is known and \( \sum_{i \sim j} \dot{x}_j(t) \) can then be approximately obtained through differentiation.

The networked multiagent system can now be described as

\[
\dot{x}(t) = - \mathcal{L}(\mathcal{G})x(t) + \gamma \mathcal{A} \dot{x}(t),
\]

\[
= -(I_n - \gamma \mathcal{A})^{-1} \mathcal{L}(\mathcal{G})x(t),
\]

\[
= - \sum_{m=0}^{\infty} (\gamma \mathcal{A})^m \mathcal{L}(\mathcal{G})x(t).
\]

Remark 3.2. Notice that the system is well-defined since \( \gamma < \frac{1}{\bar{d}_i}, (\gamma \mathcal{A})^m \to 0 \) as \( m \to \infty \). Furthermore, \((I_n - \gamma \mathcal{A})^{-1}\) exist since it follows from Lemma 2.1 that its eigenvalues lie in the strict right hand complex plane.

Equivalently, the system can be normalized by considering the following normalized consensus protocol

\[
u_i(t) = - \frac{1}{\bar{d}_i} \sum_{i \sim j} (x_i(t) - x_j(t)) + \frac{\bar{\gamma}}{\bar{d}_i} \sum_{i \sim j} \dot{x}_j(t),
\]

where \( \bar{\gamma} \in (0, 1) \). The networked multiagent system can be described using the normalized Laplacian and adjacency matrices as

\[
\dot{x}(t) = - \sum_{m=0}^{\infty} (\bar{\gamma} \bar{\mathcal{A}})^m \bar{\mathcal{L}}(\bar{\mathcal{G}})x(t).
\]

For easy of exposition we define

\[
\bar{\mathcal{L}}(\bar{\mathcal{G}}) = \sum_{m=0}^{\infty} (\bar{\gamma} \bar{\mathcal{A}})^m \bar{\mathcal{L}}(\bar{\mathcal{G}}).
\]

A. Eigenvalues

Lemma 3.1. \( \lambda_i(\bar{\mathcal{L}}(\bar{\mathcal{G}})) = \frac{1 - \lambda_i(\bar{\mathcal{A}}(\bar{\mathcal{G}}))}{1 - \bar{\gamma} \lambda_i(\bar{\mathcal{A}}(\bar{\mathcal{G}}))} \), \( i \in \{1, \ldots, n\} \).

Proof. Let \( v_i \) be the eigenvector corresponding to \( \lambda_i(\bar{\mathcal{A}}(\bar{\mathcal{G}})) \). Therefore,

\[
\sum_{m=0}^{\infty} (\bar{\gamma} \bar{\mathcal{A}})^m \bar{\mathcal{L}}(\bar{\mathcal{G}})v_i = \sum_{k=0}^{\infty} (\bar{\gamma} \bar{\mathcal{A}})^m (1 - \lambda_i(\bar{\mathcal{A}}))v_i = (1 - \lambda_i(\bar{\mathcal{A}}(\bar{\mathcal{G}}))) \sum_{k=0}^{\infty} (\bar{\gamma} \lambda_i(\bar{\mathcal{A}}))^m v_i = 1 - \lambda_i(\bar{\mathcal{A}}) \frac{1}{1 - \bar{\gamma} \lambda_i(\bar{\mathcal{A}})}v_i
\]

since \( \bar{\gamma} \lambda_i(\bar{\mathcal{A}}) < 1 \). Furthermore,

\[
\frac{1 - \lambda_i(\bar{\mathcal{A}})}{1 - \bar{\gamma} \lambda_i(\bar{\mathcal{A}})} < \frac{1 - \lambda_{i+1}(\bar{\mathcal{A}})}{1 - \lambda_{i+1}(\bar{\mathcal{A}})}
\]

since

\[
1 < \frac{1 - \bar{\gamma} \lambda_{i+1}(\bar{\mathcal{A}})}{1 - \lambda_{i+1}(\bar{\mathcal{A}})} < \frac{1 - \lambda_{i+1}(\bar{\mathcal{A}})}{1 - \lambda_{i+1}(\bar{\mathcal{A}})}.
\]

Therefore, order of the eigenvalues is preserved. This completes the proof.

Corollary 3.1. If \( \bar{\lambda}_n(\bar{\mathcal{L}}(\bar{\mathcal{G}})) \), then \( \frac{\bar{\lambda}_n(\bar{\mathcal{L}}(\bar{\mathcal{G}}))}{\lambda_n(\mathcal{L}(\mathcal{G}))} < \frac{\lambda_n(\mathcal{L}(\mathcal{G}))}{\bar{\lambda}_n(\mathcal{L}(\mathcal{G}))} \)

and \( \frac{\lambda_n(\mathcal{L}(\mathcal{G}))}{\bar{\lambda}_n(\mathcal{L}(\mathcal{G}))} \to 1 \) as \( \bar{\gamma} \to 1 \).

Proof. This is a direct result of Lemma 3.1.

Remark 3.3. Notice that \( \lambda_i(\bar{\mathcal{L}}(\bar{\mathcal{G}})) \in [0, \frac{2}{\bar{d}_i}] \). Therefore, the upper bound of possible eigenvalues of the system is dictated by \( \bar{\gamma} \). Furthermore, as \( \lambda_i(\bar{\mathcal{L}}(\bar{\mathcal{G}})) \to 1 \) as \( \bar{\gamma} \to 1 \) if \( \lambda_i(\bar{\mathcal{L}}(\bar{\mathcal{G}})) \neq 0 \). Thus, all nonzero eigenvalues of the modified Laplacian matrix approach a common value as \( \bar{\gamma} \to 1 \). The implications of this property on the condition number and the perceived graph structure of the networked multiagent system will be investigated in the following sections.

B. Condition Number

Lemma 3.2. If \( \bar{\mathcal{L}}(\bar{\mathcal{G}}) \) is undirected and balanced then \( \bar{\mathcal{L}}(\bar{\mathcal{G}}) \) is normal.

Proof. First consider that \( \bar{\mathcal{L}}(\bar{\mathcal{D}}) = \Delta(\bar{\mathcal{D}})^{-1} \mathcal{L}(\bar{\mathcal{D}}) = \frac{1}{d} \mathcal{L}(\bar{\mathcal{D}}) = \frac{1}{d} \mathcal{L}(\bar{\mathcal{D}})^T = \bar{\mathcal{L}}(\bar{\mathcal{D}})^T \) where \( d \) is the common degree among the nodes. Likewise, \( \bar{\mathcal{A}}(\bar{\mathcal{D}}) = \bar{\mathcal{A}}(\bar{\mathcal{D}})^T \).
Next consider that $-\sum_{k=0}^{\infty} (\bar{\gamma} \bar{A})^m \bar{L}(G)$ is a sum of matrices. Using the fact that a sum of normal matrices is normal if the matrices commute, it is enough to show
\begin{equation}
(\bar{\gamma} \bar{A})^p \bar{L}(\bar{\gamma} \bar{A})^m = (\bar{\gamma} \bar{A})^m \bar{L}(\bar{\gamma} \bar{A})^p, \quad \forall m, p \in \mathbb{N}
\end{equation}
and
\begin{equation}
(\bar{\gamma} \bar{A})^p \bar{L}(\bar{\gamma} \bar{A})^m = \bar{L}(\bar{\gamma} \bar{A})^m (\bar{\gamma} \bar{A})^p \bar{L}, \quad \forall m, p \in \mathbb{N}.
\end{equation}

Let $v_i$ be the eigenvector corresponding to $\lambda_i(\bar{A}(G))$,
\begin{equation}
(\bar{\gamma} \bar{A})^p \bar{L}(\bar{\gamma} \bar{A})^m v_i = (\bar{\gamma} \bar{A})^m \bar{L}(\bar{\gamma} \bar{A})^p v_i \\
\bar{A}^p \bar{L}(\lambda_i(\bar{A}))^m v_i = \bar{A}^m \bar{L}(\lambda_i(\bar{A}))^p v_i \\
(1+\lambda_i(\bar{A}))^2(\lambda_i(\bar{A}))^{m+p} v_i = (1+\lambda_i(\bar{A}))^2(\lambda_i(\bar{A}))^{m+p} v_i.
\end{equation}

Similarly,
\begin{equation}
(\bar{\gamma} \bar{A})^p \bar{L}(\bar{\gamma} \bar{A})^m v_i = \bar{L}(\bar{\gamma} \bar{A})^m (\bar{\gamma} \bar{A})^p \bar{L} v_i \\
\bar{A}^p \bar{L}(\lambda_i(\bar{A}))^m v_i = \bar{A}^m \bar{L}(\lambda_i(\bar{A}))^p v_i \\
(1+\lambda_i(\bar{A}))^2(\lambda_i(\bar{A}))^{m+p} v_i = (1+\lambda_i(\bar{A}))^2(\lambda_i(\bar{A}))^{m+p} v_i.
\end{equation}

This completes the proof.

**Corollary 3.2.** If $\bar{L}(G)$ is undirected and balanced and $1 < \hat{\kappa}(\bar{L}(G))$, then $\hat{\kappa}(\bar{L}(G)) < \hat{\kappa}(\bar{L}(\bar{G}))$ and $\hat{\kappa}(\bar{L}(\bar{G})) \to 1$ as $\gamma \to 1$.

**Proof.** This is a direct result of Corollary 3.1 and Lemma 3.2.

**Remark 3.4.** The change in the condition number of the graph is calculable given $\lambda_2(\bar{L}(\bar{G}))$, $\lambda_n(\bar{L}(\bar{G}))$ and $\gamma$.

**Remark 3.5.** Corollaries 3.1 and 3.2 illustrate the benefit of utilizing the proposed consensus protocol. In either case a relative increase of the graph’s smallest eigenvalue is possible. If the graph is balanced, a decrease in $\hat{\kappa}(\cdot)$ results in a relative increase of the system’s rate of convergence without increasing the maximum singular value of the Laplacian of the system. Therefore, a relative increase in the convergence rate is possible with a fixed graph structure. This departs from the traditional notion that rate of convergence is solely a function of the system’s Fiedler eigenvalue.

**C. Graph Structure**

In the previous sections we have shown that the proposed consensus protocol modifies the system eigenvalues. Intuitively, this change implies that the dynamical structure of the system is also changed. In order to investigate this change let
\begin{equation}
M = \sum_{k=0}^{\infty} (\bar{\gamma} \bar{A})^m
\end{equation}
and the inverse of the product of the out-degree of each node in the path sequence. Intuitively, a relatively small $[M]_{ik}$ indicate that nodes $i$ and $k$ are not strongly associated. Note that $[M]_{ii} > 1$, $[M]_{ik} > 0$ and $[M]_{ik} = [M]_{ki} \forall i, k \in V_G$ since the system is undirected and connected.

The structure of $\bar{L}(G)$ is directly computable as
\begin{equation}
\bar{L}(G)_{ij} \triangleq \begin{cases} 
[M]_{ij} = \sum_{k,i \sim k} \frac{1}{d_{out}(k)} [M]_{ik}, & \text{if } i = j, \\
[M]_{ij} = \sum_{k,j \sim k} \frac{1}{d_{out}(k)} [M]_{ik}, & \text{otherwise}.
\end{cases}
\end{equation}

Note that
\begin{equation}
\frac{\bar{\gamma}}{d_{out}(k)} [M]_{ik} = \sum_{m=0}^{\infty} \sum_{s=1}^{\infty} \bar{\gamma}^{m+1} W_{i \rightarrow k}^m \bar{A} W_{k \rightarrow p}^s (\bar{A})
\end{equation}
where $k \sim p$. Therefore, $\frac{\bar{\gamma}}{d_{out}(k)} [M]_{ik}$ is the sum of all walks of length $m+1$ composed of a walk from node $i$ to node $k$ of $m$ length and a walk from node $k$ to any neighboring node $p$ of length 1. As a result,
\begin{equation}
\sum_{k,j \sim k} \frac{1}{d_{out}(k)} [M]_{ik} = \frac{1}{\bar{\gamma}} [M]_{ij}
\end{equation}
and
\begin{equation}
\sum_{k,i \sim k} \frac{1}{d_{out}(k)} [M]_{ik} = \frac{1}{\bar{\gamma}} ([M]_{ii} - 1).
\end{equation}

The structure of $\bar{L}(G)$ can equivalently be given as
\begin{equation}
\bar{L}(G)_{ij} \triangleq \begin{cases} 
1 - \left(\frac{\bar{\gamma} - 1}{\bar{\gamma}}\right) ([M]_{ii} - 1), & \text{if } i = j, \\
\frac{\bar{\gamma} - 1}{\bar{\gamma}} [M]_{ij}, & \text{otherwise}.
\end{cases}
\end{equation}

**Theorem 3.1.** Consider a connected, undirected networked multiagent system with agent dynamics given by (8) subject to the consensus protocol given by (13). The networked multiagent system is equivalent to a network multiagent system with a weighted complete graph.

**Proof.** This is a direct result of Lemma 3.1 and equation (24).

**IV. COMPARATIVE SIMULATIONS**

In this section the proposed consensus protocol (13) and the standard consensus protocol given by (9) are compared in two examples. When the proposed consensus protocol was used $\bar{\gamma} = 0.7$. Simulations were executed at 100Hz and agents had access to the current relative state information and the neighboring velocity information from the previous time step. The rates of convergence are compared with the Laplacian potentials of the systems defined as
\begin{equation}
\Phi_G = x^T L x = \sum_{i \sim j} (x_i - x_j)^2.
\end{equation}
Note that $\Phi_G$ measures a system’s proximity to achieving consensus and $\Phi_G = 0$ if and only if the system has achieved consensus.
A. Example 1

For the first case, the networked multiagent system consists of 4 agents with a normalized adjacency matrix given by

\[ \bar{A}(\mathcal{G}) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.5 & 0 & 0.5 \\
0 & 0 & 1 & 0
\end{bmatrix}. \]

The eigenvalues of the normalized Laplacian \( \tilde{L}(\mathcal{G}) \) are given as \( \lambda_i(\tilde{L}(\mathcal{G})) \in \{0, 0.5, 1.5, 2\} \). When \( \gamma = 0.7 \), the system subject to the proposed consensus protocol can be described by the Laplacian given by (24) as

\[ \tilde{L}^e(\mathcal{G}) = \begin{bmatrix}
0.823 & -0.506 & -0.235 & -0.082 \\
-0.253 & 0.706 & -0.335 & -0.117 \\
-0.117 & -0.335 & 0.706 & -0.253 \\
-0.082 & -0.235 & -0.506 & 0.823
\end{bmatrix}. \]

where \( \lambda_i(\tilde{L}^e(\mathcal{G})) \in \{0, 0.77, 1.11, 1.18\} \). Notice that \( \kappa(\tilde{L}(\mathcal{G})) = 4 \) and \( \kappa(\tilde{L}^e(\mathcal{G})) = 1.53 \). Therefore, the proposed consensus protocol results in a faster convergence rate and a smaller condition number. Figure 1 displays the response of both systems. Notice that the proposed consensus protocol results in a visibly faster convergence rate. Figure 2 compares the Laplacian potential, \( \Phi_\mathcal{G} \), and the maximum control input norm, \( \max_{i \in \mathcal{V}_\mathcal{G}}(\|u_i\|) \), of the two systems as functions of time. Despite a faster initial decrease in the Laplacian potential of the system using the standard consensus protocol, the proposed consensus protocol converged faster to zero. Furthermore, notice that \( \max_{i \in \mathcal{V}_\mathcal{G}}(\|u_i\|) \) was similar for both systems. Thus, the faster convergence rate cannot be attributed to larger control inputs. The initial control input oscillations present with the proposed consensus protocol can be remedied with a low pass control input filter.

\[ \Delta \]

![Fig. 1. Overlapping system responses (dotted lines indicate neighbors, "X" indicate standard consensus protocol and "O" indicate proposed consensus protocol).](image1)

![Fig. 2. The Laplacian potential, \( \Phi_\mathcal{G} \), and the maximum control input norm, \( \max_{i \in \mathcal{V}_\mathcal{G}}(\|u_i\|) \) of the two systems as functions of time (dotted lines indicate standard consensus protocol).](image2)

![Fig. 3. Overlapping system responses (dotted lines indicate neighbors, "X" indicate standard consensus protocol and "O" indicate proposed consensus protocol).](image3)

B. Example 2

The number of agents is now increased to 10 and the adjacency matrix of the new system is given by

\[ \bar{A}(\mathcal{G}) = \begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0.5 & 0 & 0.5 & 0 & \cdots & 0 \\
0 & 0.5 & 0 & 0.5 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \]

The eigenvalues of the normalized Laplacian \( \tilde{L}(\mathcal{G}) \) are given as \( \lambda_i(\tilde{L}) \in \{0, 0.06, 0.23, 0.50, 0.83, 1.17, 1.50, 1.77, 1.94, 2\} \) When \( \gamma = 0.7 \), \( \lambda_i(\tilde{L}^e) \in \{0, 0.18, 0.51, 0.77, 0.94, 1.05, 1.11, 1.15, 1.18, 1.17\} \). Notice that \( \kappa(\tilde{L}(\mathcal{G})) = 33.2 \) and \( \kappa(\tilde{L}^e(\mathcal{G})) = 6.7 \). As before, the proposed consensus protocol results in a faster convergence rate and a smaller condition number. Figure 3 displays the
response of both systems. As before, the proposed consensus protocol results in a visibly faster convergence. Figure 4 compares the Laplacian potential, $\Phi_G$, and the maximum control input norm, $\max_{i \in V_G}(\|u_i\|)$, of the two systems as functions of time. The proposed consensus protocol has a much more noticeable effect on the convergence rate on the system than in Example 1. In addition, as before, $\max_{i \in V_G}(\|u_i\|)$ was similar for both systems.

V. Conclusion

We introduced and analyzed a new consensus architecture that contributes to previous studies of networked multiagent systems. Specifically, it consists of a standard term capturing relative position information and a new term capturing neighboring velocity information. We showed that addition of the latter term increases the relative convergence rate of the overall system while maintaining a fixed graph structure and without increasing the maximum eigenvalue value of the graph Laplacian. In addition, we further showed that a connected and undirected graph topology acts as a weighted complete graph topology when the proposed consensus protocol is used. Future research will include extensions to dynamic graphs to address information link failures and communication dropouts.

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